SOME PERIODIC AND NON-PERIODIC

PROBLEMS OF INTERPOLATION

by

JAGDISH PRASAD

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "SOME POISED AND NON-POISED PROBLEMS OF INTERPOLATION", submitted by Jagdish Prasad in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

We say that an interpolation problem corresponding to an incidence matrix E_n^k with k rows and n columns is poised if for any given k distinct real nodes, a unique algebraic polynomial of degree $n-1$ exists satisfying n given data on the k-nodes. In Chapter I, we generalize a result of Prof. I.J. Schoenberg and show that all interpolation problems corresponding to A-H-B matrices, weakly quasi-Hermite matrices and conservative matrices are poised if and only if the matrices satisfy Polya condition. We also show that q-H $\subset$ A-H-B $\subset$ weakly q-H = conservative matrices. A simple lemma that the horizontal union of poised matrices is poised, plays a fundamental role in the proof. Some general examples of non-conservative matrices which are poised have also been discussed.

In Chapter II and III we take $k = n$ and consider the $n \times 2n$ matrix E_n^{2n} whose first and third columns consists of 1's and all other entries are zeros. The interpolation problem corresponding to this matrix is the one initiated by P. Turán and J. Balázs as the so-called (0,2) interpolation problem and is not poised. In Chapter II we take the nodes to be the zeros of Legendre polynomials and in Chapter III we take the nodes to be the zeros of Jacobi polynomials. We obtain in each case the explicit forms of the interpolatory polynomials and study their convergence properties.

Lastly Chapter IV is devoted to the study of an interpolatory problem which seems to have been first studied by J. Balázs. Here we take the nodes to be the zeros of Jacobi polynomials $P_n^{(\alpha,-\alpha)}(x)$ and obtain a convergence theorem. For $\alpha = 0$, our result is stronger than that of J. Balázs, since under weaker hypotheses we are able to obtain the convergence for a bigger class of functions.

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INTRODUCTION

1. Let n and k be natural numbers and let $E_n^k = (\varepsilon_{ij})$ $(i = 1,2, \ldots, k;\ j = 0,1, \ldots, n-1)$ be a matrix with k rows and $n (n \geq k)$ columns having elements $\varepsilon_{ij} = 0$ or 1, which are such that $\sum_{i,j} \varepsilon_{ij} = n$ and no row is entirely composed of zeros. Let

$$(1.1) \qquad x_1 < x_2 < \ldots < x_k$$

be increasing reals and $e_n^k = \{(i,j) | \varepsilon_{ij} = 1\}$. The reals x_i and the "incidence matrix" E_n^k describe the interpolation problem

$$(1.2) \qquad P^{(j)}(x_i) = y_i^{(j)} \ , \quad \text{for} \quad (i,j) \in e_n^k$$

where $y_i^{(j)}$ are prescribed and the problem is to find the polynomial $P(x)$ of degree $\leq n-1$, which satisfies the condition (1.2). If $y_i^{(j)} = 0$ for $(i,j) \in e_n^k$ then the problem (1.2) is the homogeneous interpolation problem. In 1906, Birkhoff [4] treated this kind of general problem of interpolation. Following Schoenberg [27(1)] we shall call this the Hermite-Birkhoff (H-B) problem.

2. Examples. Some special cases of this problem are given by the following.

(1) Lagrange Interpolation: In this case we have $\varepsilon_{io} = 1$, $i = 1,2, \ldots k$; $\varepsilon_{ij} = 0$, $j \geq 1$, $i = 1,2, \ldots k$.

(2) Hermite Interpolation: If $q_1, q_2, \ldots q_k$ are integers ≥ 1, $\sum_{j=1}^{k} q_j = n$ and $\varepsilon_{ij} = 1$, $j = 0, 1, \ldots q_i-1$, $\varepsilon_{ij} = 0$, $j \geq q_i$, $i = 1,2, \ldots, k$, we have Hermite interpolation.

(3) Abel-Gontcharoff Interpolation [5]: In this case we have $\varepsilon_{ii} = 1$ and $\varepsilon_{ij} = 0$, $i \neq j$.

(4) Quasi-Hermite Interpolation (q-H) (Schoenberg [27(1)]): In this case we require for $i = 2,3, \ldots, k-1$ that $\varepsilon_{ij} = 1$, implies $\varepsilon_{ij'} = 1$ $0 \leq j' < j$.

(5) Lidstone Interpolation: In this case $k = 2$, $n = 2m$ and

$$\varepsilon_{ij} = \begin{cases} 1 & \text{for } i = 1,2 \; ; \quad j = 0,2, \ldots 2m-2 \\ 0 & \text{for } i = 1,2 \; ; \quad j = 1,3, \ldots 2m-1 \; . \end{cases}$$

(6) Taylor Interpolation: Here $k = 1$ and $\varepsilon_{ij} = 1$ for $j = 0,1, \ldots, n-1$.

(7) Generalized Lidstone Interpolation (Poritsky [22]): In this case $n = mk$ and

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } i = 1,2, \ldots, k \; ; \quad j = \ell k, \quad \ell = 0,1, \ldots, m-1 \\ 0 & \text{if } i = 1,2, \ldots, k \; ; \quad j \neq \ell k, \quad \ell = 0,1, \ldots, m-1. \end{cases}$$

(8) Generalized Abel-Gontcharoff Interpolation (Németh [16]): Here if $p_0, p_1, \ldots, p_k$ are positive integers, $p_0 + p_1 + \ldots + p_m = P(m)$, $P(-1) = 0$, $P(k) = n$, then we get a generalized Abel-Gontcharoff interpolation by considering the matrix E_n^n where

$$\varepsilon_{ij} = \begin{cases} 1 & \text{for } i = P(m-1) + 1, \ldots, P(m) \text{ and } j = P(m-1) \\ 0 & \text{for } i = P(m-1) + 1, \ldots, P(m) \text{ and } j = P(m-1) + 1, \ldots, P(m) \end{cases}$$

$$m = 0,1, \ldots, n-1.$$

3. Poised Interpolation Problems. Set

$$m_p = \sum_{i=1}^{k} \varepsilon_{ip} \quad \text{and} \quad M_p = \sum_{j=0}^{p} m_j, \quad p = 0,1, \ldots n-1, \quad M_{n-1} = n \; .$$

Definition 1. A matrix E_n^k has the Polya property (or property P) if

(3.1) $M_p \geq p+1$ for all p, $p = 0,1,\ldots,n-2$.

An interpolation problem (or equivalently the matrix E_n^k) is said to be n-poised if the problem (1.2) has a unique solution for all choices of $x_1, x_2, \ldots, x_k$ satisfying (1.1). In other words a matrix E_n^k is not poised if there exists a set of k distinct real numbers $x_1 < x_2 < \ldots < x_k$ for which the homogeneous interpolation problem corresponding to the matrix E_n^k has a non trivial solution. The necessary and sufficient condition that the interpolation problem (1.2) be poised is that

(3.2) $$\Delta = \left\| \det \frac{x_i^{\nu-j}}{(\nu-j)!} \right\| \neq 0 ,$$

for every set of real numbers satisfying (1.1), where $\nu = 0, 1, \ldots, n-1$ indicates a column and to each $(i,j) \in e_n^k$ corresponds a row of the determinant. The non-vanishing of Δ is a necessary and sufficient condition for (1.2) to be poised even when the x_i's are complex numbers. The interpolation problem corresponding to the matrix E_3^3 defined by

$$E_3^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is not poised. For if $x_2 = \frac{1}{2}(x_1 + x_3)$ then the quadratic polynomial $P(x) = (x-x_1)(x-x_3)$ satisfies all the conditions of the homogeneous interpolation problem (1.2) without vanishing identically. All the examples in §2 except (4) are poised interpolation problems. For (4) Schoenberg [27(1)] proved the following:

Theorem 1. A q-H interpolation problem is n-poised if and only if E_n^k has property P.

Theorem 1 is a generalization of result of Polya [18] who takes $k = 2$.

Many authors [27(2)], [37] have considered the problem of Lidstone interpolation. The Lidstone series for a function $f(z)$ has the form

$$f(z) = f(1)\,\Lambda_0(z) + f(0)\,\Lambda_0(1-z) + f_1''(1)\,\Lambda_1(z) + f''(0)\,\Lambda_1(1-z) + \dots \tag{3.3}$$

where $\Lambda_n(z)$ denotes the polynomial of degree $2n+1$ determined by the relations

$$\begin{aligned} &\Lambda_0(z) = z, \quad \Lambda_n(0) = \Lambda_n(1) = 0 \\ &\Lambda_n''(z) = \Lambda_{n-1}(z) \end{aligned} \qquad (n = 1,2,\dots). \tag{3.4}$$

Widder [37] has established a necessary and sufficient condition for the representation of a real function by means of an absolutely convergent Lidstone series. Since we are not concerned in the present work with the problem of representation of an entire function by a sequence of polynomials corresponding to a given poised matrix, we shall not dwell on the subject any more.

It appears that close connection exists between these problems and some results of [12], [13], and [14], but we do not wish to pursue this relationship here for want of specific results.

4. Non-poised interpolation problems. A study of interpolation problems from a different point of view was first initiated by P. Turán and his associates [31], [2] in the special case when the values and the second derivatives are prescribed on the zeros of $\pi_n(x) = (1-x^2)\,P'_{n-1}(x)$, where $P_{n-1}(x)$ is the Legendre polynomial of degree $n-1$. More precisely they solved the interpolation problem described by the following incidence matrix

$$E_{2n}^n = \begin{pmatrix} 1 & 0 & 1 & 0 & . & . & . & 0 \\ 1 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 0 & 1 & 0 & . & . & . & 0 \end{pmatrix}$$

where $k = n$ and $x_1, x_2, \ldots x_n$ are the zeros of $\pi_n(x)$. P. Turán called this type of interpolation (0,2) interpolation. It turns out that (0,2) is an example of non-poised interpolation problem. They also resolved the problems of explicit representation and of uniform convergence [2(1,2)]. As an application of this type of interpolation they established the following results [2(3)]:

Theorem 2. *Let* n *be even. If for a polynomial* $Q_{2n-1}(x)$ *of degree* $\leq 2n-1$,

$$(4.1) \qquad |Q_{2n-1}(x_\nu)| \leq A, \quad |Q''_{2n-1}(x_\nu)| \leq B, \quad \nu = 1, 2, \ldots, n,$$

then for $-1 \leq x \leq 1$ *we have*

$$(4.2) \qquad |Q_{2n-1}(x)| \leq \pi^6 \, n \, A + \frac{\pi^5 B}{n}$$

and

$$(4.3) \qquad |Q'_{2n-1}(x)| \leq \pi^8 \, n^{5/2} A + \pi^5 \, n^{1/2} B.$$

To get a proper perspective of this theorem we observe that on applying Markov's inequality to (4.2) we have $|Q'_{2n-1}(x)| \leq \pi^6 n^3 A + \pi^5 n B$ which shows that (4.3) gives a better inequality than the above.

The idea of (0,2) interpolation was extended to (0,1,3) and (0,1,2,4) by A. Sharma and his associates [26(1)],[25(3)]. For the complete literature on this type of work we shall give a chart below, using the following notation:

A: The problem of existence and uniqueness

B: The problem of explicit representation

C: The problem of uniform convergence

D: Application to inequalities as in Theorem 2.

Interpolation	Nodes	Problems	References
(0,2)	Zeros of $\pi_n(x)$, n even	A, B, C and D	[31],[2(1,2,3)]
(0,2)	Zeros of $P_n^{(\lambda)}(x)$, n even $(\lambda > -\frac{1}{2})$	A	[31]
(0,1,3)	Zeros of $\ddot{\pi}_n(x)$, n even	A, B and C	[26(1,2)]
(0,2) and (0,1,3)	Zeros of $H_n(x)$, n even	A and B	[11]
Modified (0,2)	Zeros of $\pi_n(x)$, n even	A, B and C	[25(2)]
(0,1,2,4)	Zeros of $\pi_n(x)$, n even	A, B and C	[25(1,3)]
Modified (0,2)	Zeros of $T_n(x)$, n even	A, B and C	[36]
(0,1,3)	Zeros of $T_n(x)$, n even	A and B	[34]
(0,2)	Zeros of $(1-x^2)P_n(x)$, n even	A, B, C and D	[21(1,2)]
(0,2)	Zeros of $P_n^{(1/2,-1/2)}(x)$	A, B and C	[35]
(0,2)	Zeros of $x\,L_n'(x)$	A and B	[20]
(0,3), (0,M) and (0,2,3)	$e^{2k\pi i/n}$, $0\leq k\leq n-1$	A, B and C	[28(2,3)]
Trig. (0,2)	$x_k = \frac{2k\pi}{n}$, $0\leq k\leq n-1$	A, B and C	[10]
Trig. (0,M)	$x_k = \frac{2k\pi}{n}$, $0\leq k\leq n-1$	A, B and C	[29]
(0,2)	Zeros of $\pi_n(x)$, n even	C	[8]

P_n, H_n, T_n, L_n are respectively the polynomials of Legendre, Hermite, Tcheleycheff and Laguerre. Also π_n was defined on p. 4.

5. An analogue of a Problem of J. Balázs.

In [3] J.Balázs investigates the interpolatory polynomial $Q_n(x)$ of degree 2n which satisfy the following conditions:

(5.1) $$Q_n(x_\nu) = y_\nu \ , \ [\rho(x)\ Q_n(x)]''_{x_\nu} = y''_\nu \ , \ \nu=1,2,\ldots, n,$$

$$\rho(x) = (1-x^2)^{(1+\alpha)/2}, \quad \alpha > -1$$

and

(5.2) $$Q_n(0) = \sum_{\nu=1}^{n} y_\nu \ell_\nu^2(0) \ ,$$

where x_ν ($\nu=1,2, \ldots,n$) are the zeros of $P_n^{(\alpha)}(x)$ ($\alpha > -1$) (the ultra spherical polynomial of degree n), y_ν , y''_ν are any preassigned values $\rho(x)$ is a weight function and $\ell_\nu(x)$ is the fundamental polynomial of Lagrange interpolation. We shall call this Balázs type interpolation. He proves the existence and uniqueness of these polynomials and establishes the following convergence theorem:

Theorem 3 (J.Balázs). Let $f(x)$ be a continuous function in $[-1,1]$ and let $f'(x) \in \mathrm{Lip}\ \mu$, $\mu > 1/2$. Further, let $y_\nu = f(x_\nu)$ and $y''_\nu = o(\sqrt{n})(1-x_\nu^2)^{(\alpha-3)/2}$ then the sequence of polynomials $Q_n(x)$ converges uniformly to $f(x)$ in $-1 + \varepsilon \leq x \leq 1-\varepsilon$, $0 < \varepsilon < 1$ (ε being arbitrary fixed postive number).

He also shows that there does not exist a polynomial $Q_n(x)$ of degree $\leq$ 2n-1 satisfying only (5.1). Therefore the condition (5.2) is necessary for the existence of polynomial $Q_n(x)$ of degree 2n. In [19(1,2)] we have also made similar investigations in the case when the nodes are the zeros of $H_n(x)$ and $L_n^{(\alpha)}(x)$ ($\alpha > -1$) , where $H_n(x)$ and $L_n^{(\alpha)}(x)$ are n^{th} Hermite and Laguerre polynomials respectively.

6. Summary of the thesis. In Chapter I we introduce the notion of horizontal union of incidence matrices E_n^k and E_m^k and define the class of simple matrices. Incidence matrices which satisfy the Polya condition or (P) condition of §3 and which can not be expressed as the union of two or more such matrices are called simple. We introduce the class of A-H-B matrices as the horizontal union of matrices each of which is q-H and satisfies (P) condition. Our results of Chapter I depend upon the fundamental lemma that the horizontal union of two poised matrices is poised. Thus all A-H-B matrices are shown to be poised which generalizes the result of Schoenberg [27(1)].

We further extend the class of A-H-B matrices by introducing simple weakly q-H matrices. This definition is rather complicated and we refer to §4 of Chapter I for details. It turns out that the class of A-H-B matrices are strictly contained in the class of weakly q-H matrices. We give a characterization of weakly q-H matrices and show that every simple weakly q-H matrix is poised.

Lastly we introduce the class of conservative matrices which coincides with the class of weakly q-H matrices so that every conservative matrix satisfying (P) condition is poised. We close the chapter by showing that there are matrices which are not conservative and are still poised. We also show that the results of Poritsky [22] about the existence and explicit forms of his interpolatory polynomials are immediate from our Theorem 1.3.2 and the fundamental lemma 1.4.1. Also the results of Németh [16] follow from these same. A few remarks on trigonometric interpolation are meant to throw some light on an area of open problems.

Chapter II and III are devoted to a study of non-poised problems. In the second chapter besides the existence theorem we establish results concerning the explicit representation anduniform convergence for interpolatory polynomial $S_n(x)$ on the zeros x_ν ($\nu=2, \ldots, n+1$) of $P_n(x)$ (where $P_n(x)$ is n^{th} Legendre polynomial) satisfying the following conditions:

$$S_n(x_\nu) = \alpha_\nu \quad \text{and} \quad S_n''(x_\nu) = \beta_\nu \; , \; (\nu=2, \ldots, n+1) \; ,$$

where α_ν and β_ν are any preassigned values. For the convergence of the sequence of polynomials $S_n(x)$ we require that $f''(x) \in \text{Lip}\, \mu$, $\mu > \frac{1}{2}$. Our theorem can be compared with that of Saxena [25(2)] who proved the convergence theorem under similar conditions.

In Chapter III we investigate the interpolatory polynomials $R_n(x)$ on the zeros x_ν ($\nu=0,1,\ldots, n+1$) of $(1-x^2)\, P_n^{(\alpha,-\alpha)}(x)$ ($\alpha > -1$) (where $P_n^{(\alpha,-\alpha)}(x)$ is n^{th} Jacobi polynomial) such that

$$R_n(x_\nu) = \alpha_\nu \; , \quad (\nu= 0,1, \ldots n+1)$$

and

$$R_n''(x_\nu) = \beta_\nu \; , \quad (\nu= 1, \ldots, n) \; .$$

In this case it is interesting to observe that the interpolatory polynomials exist for both n odd and even. We have succeeded in proving the uniform convergence only for $\alpha = \frac{1}{3}$, although the existence theorem holds for $0 \leq \alpha < 1$. For $\alpha = \frac{1}{3}$, $R_n(x;f)$ converges to $f(x)$ uniformly for $-1 + \varepsilon \leq x \leq 1-\varepsilon$, $0 < \varepsilon < 1$ (ε being an arbitrary fixed positive number) when $f(x) \in \text{Lip}\, \mu$ ($\mu > \frac{1}{6}$) . Elsewhere [21(1)] we have shown that if $\alpha = 0$ and n is even then $R_n(x;f)$ converges to $f(x)$ uniformly on $[-1,1]$ when $f(x)$ satisfies the Zygmund condition

$$f(x+h) - 2f(x) + f(x-h) = o(h)$$

in $[-1,1]$ and $|\beta_\nu| \leq \dfrac{o(h)}{(1-x_\nu^2)^{1/2}}$. We could not prove the uniform convergence theorem for other values of α due to the fact that we come across a hypergeometric function for which a simple value seems to be known only for $\alpha = \frac{1}{3}$ and $\alpha = 0$.

Chapter IV deals with the study of Balázs type interpolation problem on the zeros of $P_n^{(\alpha,-\alpha)}(x)$. We investigate the polynomials $R_n(x)$ of degree $2n$ which satisfy conditions (5.1) and (5.2) with weight function $\rho(x) = (1-x)^{(1+\alpha)/2} (1+x)^{(1-\alpha)/2}$ $(\alpha > -1)$. We prove that there exists a unique polynomial of degree $2n$ for all $n \geqslant 1$ and also prove the uniform convergence theorem under the condition that $f'(x) \in \text{Lip}\ \mu$, $\mu > 0$. A comparison of our convergence Theorem 4.2.2 with that of Balázs in §5 above shows that under the weaker condition $y''_\nu = o(n^{3/4})(1-x_\nu)^{(\alpha-3)/2}(1+x_\nu)^{-(\alpha+3)/2}$ we have a larger class of functions for which uniform convergence is available.

Remark. It has been brought to my notice that some interesting results and conjectures on the subject of (0,2) interpolation on equidistant nodes have been obtained by T.S. Motzkin and J. Dyer. Since no reference to this work has been available it has not been possible to report on them here.

CHAPTER I

ABEL-HERMITE-BIRKHOFF INTERPOLATION

1. Introduction. Recently I.J. Schoenberg [27(1)] has given an interesting generalization of a theorem of Polya [18] which deals with an interpolation problem on two nodes. If $x_1 < x_2 < \ldots < x_k$ are k nodes, I.J. Schoenberg introducrs the term "quasi-Hermite" interpolation when the interpolation is of Hermite type on all points except perhaps x_1 and x_k. His result gives a necessary and sufficient condition for a quasi-Hermite problem to be posied. His method of proof depends on the existence and uniqueness of Abel-Gontcharoff interpolation, which is not itself of quasi-Hermite type. The object of this chapter is to generalize the result of Schoenberg and to define certain classes of poised interpolation problems. We have not succeeded in characterizing all the poised interpolation problems, but by introducing the notions of A-H-B (Abel-Hermite-Birkhoff) matrices, weakly q-H matrices and conservative matrices, we are able to characterize large classes of poised problems of interpolation.

In §2 we give the preliminaries and definitions. §3 is devoted to a study of A-H-B matrices. In §4, we introduce the class of weakly quasi-Hermite matrices and prove our main result. In §5, we introduce the notion of conservative matrices and §6 is devoted to the proof of some theorems for matrices which are neither weakly q-H nor conservative. In §6, we also formulate a conjecture based on our earlier theorems. We devote §7 to show how the results of Poritsky [22] and Németh [16] are special cases of our theorems. Lastly in §8, we make some remarks on trigonometric interpolation but it appears our results do not extend naturally to this case or to general Tchebycheff systems [30].

2. Preliminaries and definitions.

Let k,n be natural numbers and let $E_n^k = (\varepsilon_{ij})(i=1,2,\ldots k;\ j=0,1,\ldots n-1)$ be (k,n) matrix where $\varepsilon_{ij} = 1$ or 0 and $\sum_{ij} \varepsilon_{ij} = n$. Let $x_1 < x_2 < \ldots < x_k$ be increasing reals and let $e_n^k = \{(i,j) | \varepsilon_{ij} = 1\}$. The real numbers $\{x_i\}_1^k$ and the incidence matrix E_n^k describe the interpolation problem (1.2). We shall suppose that our given matrix E_n^k has no row consisting only of zeros [In the next section we shall consider the decomposition of E_n^k into submatrices which will not be required to satisfy this requirement].

<u>Definition</u>. <u>For any natural number</u> $s \geq 1$, <u>the matrix</u> E_s^k <u>will be called</u> q-H <u>if it has the following property</u>:

(1.2.1) $\qquad \varepsilon_{ij} = 1,\ \ i = 2, \ldots, k-1$ <u>implies</u> $\varepsilon_{ij'} = 1$ for $0 \leq j' < j$.

We refer to §1 of the introduction for the definition of the Polya property.

Set $Z_p = \{x_i | \varepsilon_{ip} = 1 \ ,\ i = 2, \ldots, k-1 \ \ p = 0,1,\ldots, n-1$.

Then an equivalent way of stating that E_n^k is q-H is that

(1.2.2) $$Z_0 \supset Z_1 \supset \ldots \supset Z_{n-1}$$

The matrix for Abel-Gontcharoff interpolation is the unit matrix and does not satisfy (1.2.1) or (1.2.2), but the problem is poised. It is easy to construct other examples of incidence matrices which do not satisfy (1.2.1), but for which the corresponding interpolation problem is poised.

For example if

$$E_4^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the determinant $\Delta = -12(x_3 - x_2) \neq 0$. So the problem in this case is poised but the matrix E_4^4 is not q-H .

3. Horizontal decomposition of matrices.

Definition. E_n^k will be called horizontal union of matrices ${}_iE_{p_i}^k$ $(i = 1,2, \ldots, r)$, symbolically

$$E_n^k = {}_1E_{p_1}^k \oplus {}_2E_{p_2}^k \oplus \ldots \oplus {}_rE_{p_r}^k , \tag{1.3.1}$$

if E_n^k can be decomposed into blocks of submatrices ${}_iE_{p_i}^k$ for which

$$M_{p_i - 1} = p_i . \tag{1.3.2}$$

It may be observed that p_i can be less than k .

If each ${}_iE_{p_i}^k$ is also minimal in the sense that it does not have further decomposition into submatrices satisfying (1.3.2), we shall say that the horizontal decomposition

$$E_n^k = {}_1E_{p_1}^k \oplus \ldots \oplus {}_rE_{p_r}^k$$

is maximal. It is easy to verify a maximal horizontal decomposition is unique. For example if

$$E_5^4 = \begin{pmatrix} 0&0&0&1&0 \\ 1&0&0&0&0 \\ 1&0&0&1&0 \\ 0&0&1&0&0 \end{pmatrix}$$

then

$$E_5^4 = \begin{pmatrix} 0&0 \\ 1&0 \\ 1&0 \\ 0&0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1&0 \\ 0&0 \\ 1&0 \\ 0&0 \end{pmatrix}$$

is a maximal decomposition while

$$E_5^4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is not a maximal decomposition. Also the operation of union is associative but not commutative.

<u>Definition</u>. <u>If</u>

$$E_n^k = {}_1E_{p_1}^k \oplus \dots \oplus {}_rE_{p_r}^k$$

$$F_n^k = {}_sE_{p_s}^k \oplus \dots \oplus {}_tE_{p_t}^k$$

<u>we shall say that</u> $E_n^k \sim F_n^k$ if $(s, \dots, t)$ <u>is a permutation of</u> $(1, \dots, r)$.

This definition gives an equivalence relation. The reflexitivity and symmetry are obvious and transitivity follows from the fact that permutation is transitive.

Thus $$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \quad \dots$$

In fact it is easy to see that among all (3,3) matrices there are 15 A-H-B matrices and when they are divided into classes by means of this equivalence relation there are seven classes which contain 1, 1, 1, 2, 2, 2, 6 elements respectively. The 7 equivalence classes in this case can be given by the following seven matrices each representing the class to which it belongs.

$$\begin{pmatrix}1&0&0\\1&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\1&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&0&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\1&0&0\end{pmatrix}.$$

Definition. A submatrix of E_n^k is called p-reduced if it is obtained from E_n^k by removing its first p columns.

Definition. E_n^k is called A-H-B if there exists an integer $r \geq 1$ such that (1.3.1) holds where ${}_iE^k_{p_i}$ is q-H for $i = 1,2, \ldots, r$.

We shall now give a characterization of A-H-B matrices.

Theorem 1.3.1. A matrix E_n^k is A-H-B if and only if the following property is true.

(Q) If for some p, $p = 0,1, \ldots, n-2$, $M_p = \alpha > p+1$ then for each i, $i = 2, \ldots, k-1$, $\varepsilon_{i,p+1} = 1$ implies $\varepsilon_{ip} = 1$.

Proof. If E_n^k is A-H-B let the maximal decomposition of E_n^k be given by

$$E_n^k = {}_1F^k_{p_1} \oplus {}_2F^k_{p_2} \oplus \ldots \oplus {}_rF^k_{p_r}, \quad p_1 + p_2 \ldots + p_r = n$$

where each of ${}_\lambda F^k_p$ is q-H and ${}_\lambda\mu_{p_\lambda -1} = p_\lambda$, for

$${}_\lambda\mu_{p_\lambda -1} = \sum \varepsilon_{ij}, \quad \varepsilon_{ij} \in {}_\lambda F^k_{p_\lambda}.$$

Let $M_q = \alpha > q+1$ for some q, $q = 0,1, \ldots, n-2$ then $q \neq p_1 + p_2 + \ldots + p_\lambda - 1$, $\lambda = 1, \ldots, r$. Let for some j, $j = 1, \ldots, r-1$

$$p_1 + p_2 + \ldots + p_j \leq q \leq p_1 + p_2 + \ldots + p_{j+1} - 2.$$

Thus the $(q-1)^{th}$ column belong to ${}_{j+1}F^k_{p_{j+1}}$ which is q-H. Hence property (Q) is true. This proves the necessity of condition (Q).

Assume that property (Q) holds. If $M_p = p+1$ for all p, $p = 0,1, \ldots, n-1$ then each column has only one non-zero element and since by hypothesis E^k_n has no rows consisting only of zeros it follows that $k = n$ and E^k_n is obtained from a column permutation of the identity matrix 1 . Hence E^k_n is A-H-B , since each column of the identity matrix is q-H .

If M_p $p+1$ for all p, $p = 0,1, \ldots, n-2$, then $M_{n-2} = M_{n-1} = n$ and therefore by property (Q) the matrix E^k_n is q-H and hence A-H-B . It therefore remains to discuss the case when there exists integers $p_1, p_2, \ldots, p_r$ such that

$$M_{P(\lambda)-1} = P(\lambda) \ , \quad \lambda = 1, \ldots, r \ , \ \sum_{i=1}^{\lambda} p_i = P(\lambda) \tag{1.3.3}$$

and

$$M_q > q+1 \quad \text{for} \quad P(\lambda) \le q \le P(\lambda+1) - 1 \ , \quad (\lambda = 1, \ldots, r-1) \ . \tag{1.3.4}$$

Because of (1.3.4) and property (Q) the matrix of columns beginning from $(p_1 + p_2 + \ldots + p_\lambda)^{th}$ to $(\sum_1^{\lambda+1} p_i=1)^{th}$ (inclusive) is a q-H matrix $(\lambda = 1, \ldots, r-1)$. Hence E^k_n in terms of q-H matrices of the required type. Hence E^k_n is A-H-B which proves the theorem.

Theorem 1.3.2. *An A-H-B matrix E^k_n corresponds to a poised interpolation problem if and only if the matrix E^k_n satisfies Polya's condition (P).*

The proof of this theorem will follow as a consequence of a more general Theorem 1.4.1 which we state and prove in §4.

Theorem 1.3.3. *If E_n^k and F_n^k are q-H matrices satisfying condition (P) and if their maximal decomposition do not have more than two components then $E_n^k \sim F_n^k$ implies $E_n^k \equiv F_n^k$.*

Proof. If E_n^k and F_n^k are q-H and if they have no components, then it is obvious that $E_n^k \sim F_n^k$ implies $E_n^k \equiv F_n^k$.

Let now $E_n^k = E_{p_1}^k \oplus E_{p_2}^k$ and $F_n^k = E_{p_2}^k \oplus E_{p_1}^k$. We shall show that if E_n^k is q-H, then F_n^k is not q-H. Since the decomposition is maximal, $M_{p_1-1} = p_1$ and therefore the last column of $E_{p_1}^k$ consists either only of zeros or has only one non-zero element. This non-zero element can not belong to Z_{p_1-1} since otherwise all the elements in the row would be 1 by q-H property and so our decomposition would not be maximal. Hence $Z_{p_1-1} = \phi$. A fortiori $Z_{p_1} = \phi$ and hence $Z_{p_2} = \phi$ by the q-H property. Hence F_n^k can not be q-H. This completes the proof of the theorem.

The theorem is not true if E_n^k and F_n^k have more than two components in their maximal decompositions. For example

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are equivalent but not congruent.

The theorem is not true if E_n^k and F_n^k are A-H-B. For example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

are A-H-B and equivalent but are not identical.

4. Weakly quasi-Hermite matrices.

For $p = 0, 1, \ldots, n-1$ we set $\overline{Z}_p = \{x_i | 1 \leq i \leq k, \varepsilon_{ip} = 1\}$ and $Z^j_{p,p+1,\ldots j+1} = \overline{Z}_j - \partial(\overline{Z}_p \cup \overline{Z}_{p+1} \cup \ldots \cup \overline{Z}_{j+1})$ where $\partial(\overline{Z}_p \cup \overline{Z}_{p+1} \cup \ldots \cup \overline{Z}_{j+1})$ are the extreme points $x_A \leq x_B$ of $\overline{Z}_p \cup \overline{Z}_{p+1} \cup \ldots \cup \overline{Z}_{j+1}$, where $A = A(p,j+1) \leq B = B(p,j+1)$. If $p = 0$ we write $A(0,j+1) \equiv A(j+1)$. Set

$$\Omega_j = (x_{A(j)}, x_{B(j)}) - \overline{Z}_j, \qquad 0 \leq j \leq n-1 .$$

<u>Definition</u>. <u>We shall say that a matrix</u> E^k_n <u>is simple if</u> E^k_n <u>has the following property</u>: <u>For all</u> j, $0 \leq j \leq n-2$, <u>either</u> $M_j < j+1$ or $M_j > j+1$.

<u>Definition</u>. <u>A simple matrix</u> E^k_n <u>is said to be weakly</u> q-H <u>if it satisfies the following conditions</u>:

(i) E^k_n <u>satisfies condition</u> (P)

(ii) <u>For every</u> j, $1 \leq j \leq n-1$ <u>and</u> $A(j-1) < i < B(j-1)$,

$\varepsilon_{ij} = 1$ <u>implies</u> $\varepsilon_{i,j-1} = 1$.

<u>Definition</u>. <u>A matrix</u> E^k_n <u>is said to be weakly</u> q-H <u>if it is the horizontal union of simple weakly</u> q-H <u>matrices</u>.

A matrix E_n^k is said to be weakly quasi-Hermite if the following property is true: If there exists integers p and q, $p < q$, $M_{p-1} = p$ and $M_j > j+1$ for all j, $p \leq j \leq q-1$ and $M_q = q+1$ then

$$Z^j_{p,p+1,\ldots j+1} \supseteq Z^{j+1}_{p,p+1,\ldots j+1} \text{ for all } j, \quad p \leq j \leq q-1 .$$

For example

$$E_5^4 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad E_6^4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are weakly q-H , but not q-H and are not A-H-B .

$$E_5^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$ is a simple weakly q-H matrix but

is not A-H-B . Another example of a simple matrix which is weakly q-H is

$$E_4^3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

We are now in a position to make the following observations:

1. Every q-H matrix is weakly q-H .
2. Every A-H-B matrix is weakly q-H .
3. Every weakly q-H matrix satisfying Polya's condition (P) can be written uniquely as the union of a finite number of simple weakly q-H matrices.

We shall prove the following result.

<u>Theorem 1.4.1.</u> <u>A weakly</u> q-H <u>matrix</u> E_n^k <u>is poised if and only if Polya's condition</u> (P) <u>is satisfied.</u>

The necessity of condition (P) has already been proved by Schoenberg and the same proof is valid in this case as well.

In order to prove the sufficiency of this condition we shall require the following lemmas.

<u>Lemma 1.4.1.</u> <u>If</u> $E_n^k = F_p^k \oplus G_r^k$ $(n = p+r)$, F_p^k , G_r^k <u>being incidence matrices of poised interpolation problems</u>, <u>then</u> E_n^k <u>defines a poised interpolation problem</u>.

<u>Proof</u>: Let $f_p^k = \{(i,j) \mid \varepsilon_{ij} = 1 , \quad \varepsilon_{ij} \in F_p^k\}$ and let g_r^k have a similar meaning with respect to G_r^k .

Since F_p^k is poised, there exists a unique polynomial $\phi(x)$ of degree $\leq$ p-1 such that

$$\phi^{(j)}(x_i) = y_i^{(j)} , \quad (i,j) \in f_p^k .$$

Similarly there exists a polynomial $\psi(x)$ of degree $\leq$ r-1 such that

$$\psi^{(j)}(x_i) = u_i^{(j)} , \quad (i,j) \in g_r^k .$$

Set $\Psi(x) = \frac{1}{(p-1)!} \int_0^x \psi(t)(x-t)^{p-1}dt$ and determine a polynomial $\Phi(x)$ of degree p-1 by the requirement that

$$\Phi^{(j)}(x_i) = y_i^{(j)} - \Psi^{(j)}(x_i) , \quad (i,j) \in f_p^k .$$

The solution of the problem of interpolation with incidence matrix E_n^k is then $\Phi(x) + \Psi(x)$.

Lemma 1.4.2. *If E_n^k is a simple weakly* q-H *matrix satisfying Polya's condition* (P) *and* $\phi(x)$ *is a function such that*

$$\phi^{(j)}(x_i) = 0 \quad \text{for} \quad (i,j) \in e_n^k$$

then $\phi^{(q)}(x)$ $(q = 0,1, \ldots, n-1)$ *vanishes in at least* $M_{q-1}-q$ *distinct points in* Ω_q *where by definition* $M_{-1} = 0$.

Remark. When E_n^k is simple weakly q-H then $A = A(q) < B = B(q)$ and $\Omega_q = (x_A, x_B) - \overline{Z}_q$.

Corollary. *Every simple weakly* q-H *interpolation problem is poised if and only if it satisfies the condition* (P).

Proof of Lemma 1.4.2. We shall prove the lemma by induction on q. If $q = 0$ then the statement of the lemma is true since $M_{-1} = 0$. Now let us assume that the statement is true for some value of q, $0 \leq q \leq n-2$ and prove the statement for $q+1$. By our induction hypothesis $\phi^{(q)}(x)$ vanishes in at least $M_{q-1} - q$ distinct points of Ω_q . Since $\phi^{(q)}(x)$ has also m_q zeros in $[x_A, x_B]$ so $\phi^{(q)}(x)$ has in $[x_A, x_B]$,

$$M_{q-1} - q + m_q = M_q - q$$

distinct zeros. Because E_n^k is simple and satisfies condition (P), we have $M_q - q \geq 2$. Therefore by Rolle's theorem $\phi^{(q+1)}(x)$ has at least $M_q - q-1$ zeros in (x_A, x_B) and these zeros are intermediate between the consecutive zeros of $\phi^{(q)}(x)$ among which are all the points of $\overline{Z}_q$. Therefore none of these $M_q - q-1$ zeros may lie in $\overline{Z}_q$. Now there are only two possibilities:

(i) $\overline{Z}_{q+1} = \phi$ or (ii) $\overline{Z}_{q+1}$ has at least one element.

If $\overline{Z}_{q+1} = \phi$ then all the points $M_q - q-1$ lie in Ω_{q+1}. Since $\Omega_{q+1} = (x_{A'}, x_{B'}) - \overline{Z}_{q+1}$, $A' = A(q+1)$ and $B' = B(q+1)$, so if $\overline{Z}_{q+1} = \phi$, $A' = A(q)$, $B' = B(q)$ and therefore $\Omega_{q+1} = (x_A, x_B)$.

From the definition of Ω_{q+1} one can see that

$$\Omega_{q+1} = (x_{A'}, x_{B'}) - \overline{Z}_{q+1} \cap (x_{A'}, x_{B'}) \text{ and } \quad (\text{since for sets } P,Q \;\; P-Q = P-P\cap Q).$$

$$\Omega_q = (x_A, x_B) - \overline{Z}_q \cap (x_A, x_B)$$

$$= (x_A, x_B) - \overline{Z}_q \cap (x_{A'}, x_{B'}) .$$

From the definition we have

$$Z^{q+1}_{0,1,\dots q+1} = \overline{Z}_{q+1} - \partial(\overline{Z}_0 \cup \overline{Z}_1 \cup \dots \cup \overline{Z}_{q+1})$$

$$= \overline{Z}_{q+1} - \{x_{A'}, x_{B'}\} \cap \overline{Z}_{q+1} .$$

Similarly

$$Z^{q}_{0,1,\dots q+1} = \overline{Z}_q - \{x_{A'}, x_{B'}\} \cap \overline{Z}_q .$$

Since E^k_n is weakly q-H therefore

$$\overline{Z}_q - \{x_{A'}, x_{B'}\} \cap \overline{Z}_q \supseteq \overline{Z}_{q+1} - \{x_{A'}, x_{B'}\} \cap \overline{Z}_{q+1}$$

which implies that

$$\overline{Z}_{q+1} \cap (x_{A'}, x_{B'}) \subseteq \overline{Z}_q \cap (x_{A'}, x_{B'})$$

and hence

$$\Omega_q \subseteq \Omega_{q+1} .$$

This completes the proof of the lemma.

<u>Remarks</u>. 1. If $M_0 = 1$, and the 1-reduced matrix is A-H-B then the problem is poised.

2. If $M_0 = \alpha > 1$, $M_0 = M_1 = M_{\alpha-1} = \alpha$ then the problem is poised if and only if α-reduced matrix is poised.

3. Following is an example of a weakly q-H matrix which does not satisfy condition (P)

$$E_4^3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an example of a q-H matrix which does not satisfy condition (P) .

5. <u>Conservative matrices</u>. We shall call a set of successive 1's occurring in a row of an incidence matrix E_n^k a <u>sequence</u> if it is neither preceded nor followed by a 1. We shall call a sequence of 1's occurring in an incidence matrix E_n^k an <u>odd</u> or <u>even</u> according as there are an odd or even number of 1's in the sequence.

<u>Definition</u>. <u>A sequence of 1's occurring in an incidence matrix E_n^k is said to be a Hermite sequence if its first element belongs to the 0^{th} column</u>.

<u>Definition</u>. <u>A sequence of 1's occurring in the i^{th} row and beginning in the j^{th} column of a simple incidence matrix E_n^k is called a conservative sequence if either</u> (i) <u>it forms a Hermite sequence</u>

<u>or</u> (ii) $x_i \notin (x_{A(j-1)}, x_{B(j-1)})$, $j > 0$, <u>where</u> $x_{A(j-1)} \le x_{B(j-1)}$ <u>are</u>

extreme points of the set $(\overline{Z}_0 \cup \overline{Z}_1 \cup \dots \cup \overline{Z}_{j-1})$.

We shall call an incidence matrix E_n^k, a conservative matrix if it is the union of simple matrices which have only conservative sequences.

Theorem 1.5.1. Every weakly q-H matrix is conservative and every conservative matrix is weakly q-H.

Proof. Suppose E_n^k is simple weakly q-H but not conservative. Then there exists a non conservative sequence of 1's beginning in the i^{th} row and j^{th} column, $2 \leq i \leq n-1$ and $j \geq 1$ and $x_i \in (x_{A(j-1)}, x_{B(j-1)})$. Since E_n^k is simple and weakly q-H therefore $\varepsilon_{ij} = 1$ for $A(j-1) < i < B(j-1)$ implies $\varepsilon_{i,j-1} = 1$ which is a contradiction since by assumption the sequence begins in the j^{th} column. So $x_i \notin (x_{A(j-1)}, x_{B(j-1)})$ and E_n^k is conservative.

An equivalent statement of definition of weakly q-H matrix E_n^k is that for every j, $1 \leq j \leq n-1$ and $\varepsilon_{ij} = 1$, $\varepsilon_{i,j-1} = 0$ implies $x_i \notin (x_{A(j-1)}, x_{B(j-1)})$. This formulation implies that if E_n^k is conservative then it is weakly q-H. Now we are in a position to state the following:

Theorem 1.5.2. Every conservative matrix E_n^k is poised if and only if it satisfies condition (P).

Theorem 1.5.2 follows from Theorems 1.5.1 and 1.4.1.

6. Some poised matrices, not weakly q-H.

If $k = 3$ and $n = 3$, then for $M_0 = 1$, the matrix is not poised if the 1-reduced matrix is not poised. But the 1-reduced matrix still satisfies Polya's condition and so is poised.

Hence if E_3^3 is not poised, $M_0 > 1$.

If $M_0 = 3$, then the only matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ which is still poised.

If $M_0 = 2$, then the only matrix which is not poised is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

which is not A-H-B. Thus if $k = 3$ and $n = 3$, the only poised matrices are the weakly q-H matrices which satisfy Polya's condition. This may lead one to believe that the only poised matrices are the conservative matrices satisfying (P). However this is not so. For example when $k = 3$ and $n = 4$

$$E_4^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is poised since in this case

$$|\Delta| = 2(x_3-x_1) \mid (x_3-x_2)^2 + (x_2-x_1)^2 - (x_2-x_1)(x_3-x_2) \mid .$$

But this matrix is not weakly q-H. In the following theorems we exhibit two classes of matrices which are poised but which are not weakly q-H. Set $(a)_p = \overbrace{a\ a\ a\ \ldots\ a}^{p \text{ times}}$. Then we have

<u>Theorem 1.6.1</u>. <u>If</u> E_n^3 <u>is given by</u>

$$E_n^3 = \begin{pmatrix} (1)_{p_1} & (0)_{p_3-1} & (1)_{q_1} & 0 \ldots 0 \\ (0)_{p_1+p_3-1} & & (1)_{q_2} & 0 \ldots 0 \\ (1)_{p_3} & (0)_{p_1-1} & (1)_{q_3} & 0 \ldots 0 \end{pmatrix}$$

where $p_1 + p_3 + q_1 + q_2 + q_3 = n$ *and* q_2 *is even, then the matrix is poised*.

Proof. The associated matrix F^3_{n-1} obtained by replacing E^3_n in the first column by $(1\ 0\ 0)^T$ is the union of two quasi-Hermite matrices satisfying Polya's condition and is therefore poised by Theorem 1.3.2. Hence there does not exist a non-trivial polynomial of degree $n-2$ satisfying the homogeneous interpolation problem of F^3_{n-1}. But there does exist a polynomial $R(x)$ of degree $n-1$ which does so. Indeed

$$(1.6.1) \qquad R(x) = \frac{a}{(p_1+p_2-2)!} \int_{x_1}^{x} (t-x_1)^{q_1}(t-x_2)^{q_2}(t-x_3)^{q_3}(x-t)^{p_1+p_3-2}\,dt + Q(x)$$

where $Q(x)$ is a polynomial of degree p_1+p_3-2. Since

$$R(x_1) = R'(x_1) = \ldots = R^{(p_1-1)}(x_1) = 0$$

$$R'(x_3) = \ldots = R^{(p_3-1)}(x_3) = 0\ ,$$

we have

$$(1.6.2) \qquad Q(x_1) = Q'(x_1) = \ldots = Q^{(p_1-1)}(x_1) = 0$$

and

$$Q'(x_3) = -\frac{1}{(p_1+p_3-3)!} \int_{x_1}^{x_3} P(t)(x_3-t)^{p_1+p_3-3}\,dt$$

$$= -I_1(x_3)\ , \text{ where } \ P(t) = a(t-x_1)^{q_1}(t-x_2)^{q_2}(t-x_3)^{q_3} \tag{1.6.3}$$

$$Q^{(j)}(x_3) = - I_j(x_3)\ , \qquad 2 \leq j \leq p_3-1\ ,$$

$$I_j(x_3) = \frac{1}{(p_1+p_3-j-2)!} \int_{x_1}^{x_3} P(t)(x_3-t)^{p_1+p_3-j-2}\, dt\ . \tag{1.6.4}$$

The polynomial $Q(x)$ is to be determined from the interpolation problem corresponding to the matrix

$$\begin{pmatrix} (1)_{p_1} & 0 \ldots & 0 \\ 0(1)_{p_3-1} & 0 \ldots & 0 \end{pmatrix}$$

which satisfies Polya's condition. We shall use a known formula generalizing the Taylor's formula and is an expansion about 2 given points. This formula seems to be due to Obreschkoff [17] (See also P. Hummel and Seebeck [9]).

$$f(x) = f(a) + \sum_{k=1} \frac{(m+n-k)!}{(m+n)!} \left\{ \binom{m}{k} f^{(k)}(a) - (-1)^k \binom{n}{k} f^{(k)}(x) \right\} \times (x-a)^k + R$$

where

$$R = (-1)^n \frac{m!\ n!\ (x-a)^{m+n+1}}{(m+n)!\ (m+n+1)!} f^{(m+n+1)}(a+\theta(x-a))\ , \quad 0 < \theta < 1\ .$$

Using this we have from (1.6.2) and (1.6.3),

$$Q(x_3) = \sum_{k=1}^{p_3-1} \frac{(p_1+p_3-2-k)!}{(p_1+p_3-2)!} (-1)^k \binom{p_3-1}{k} I_k(x_3)(x_3-x_1)^k\ .$$

If $R(x_3) = 0$ then $R(x)$ satisfies the homogeneous interpolation problem of E_n^3. Now $R(x_3) = 0$ implies

$$\frac{1}{(p_1+p_2-2)!} \int_{x_1}^{x_3} P(t)(x_3-t)^{p_1+p_3-2}\, dt \tag{1.6.5}$$

$$+ \sum_{k=1}^{p_3-1} \frac{(p_1+p_3-2-k)!}{(p_1+p_3-2)!} (-1)^k \binom{p_3-1}{k} I_k(x_3)(x_3-x_1)^k = 0$$

From (1.6.5) and (1.6.4) we have

$$\frac{1}{(p_1+p_2-2)!}\Bigg[\int_{x_1}^{x_3} P(t) \sum_{k=1}^{p_3-1} (-1)^k \binom{p_3-1}{k} (x_3-x_1)^k \times (x_3-t)^{p_1+p_3-2-k} dt + \int_{x_1}^{x_3} P(t)(x_3-t)^{p_1+p_3-2} dt \Bigg]$$

$$= 0$$

whence on simplification we have

$$\int_{x_1}^{x_3} P(t)(x_3-t)^{p_1-1} (t-x_1)^{p_3-1} dt = 0 \quad .$$

Hence $a = 0$ so that E_n^3 is poised, which completes the proof of the theorem.

Similarly we can prove the following

<u>Theorem 1.6.2.</u> <u>Let</u> p_1, p_3, q_2, r_1, s_1, s_2, s_3 <u>be non-negative integers satisfying the conditions</u>

$$p_1+p_3+q_2+r_1+\sum_{i=1}^{3} s_i = n \ , \quad p_1+p_3 \geq 1, \ q_2 \ \underline{\text{and}} \ s_2 \ \underline{\text{even}} \ .$$

<u>Then the incidence matrix</u> E_n^3 <u>given by</u>

$$E_n^3 = \begin{pmatrix} (1)_{p_1} & (0)_{p_3-1+q_2} & (1)_{r_1+s_1} & 0 \ldots\ldots 0 \\ (0)_{p_1+p_3-1} & (1)_{q_2} & (0)_{r_1}(1)_{s_2} & 0 \ldots\ldots 0 \\ (1)_{p_3} & (0)_{p_1-1+q_2+r_1} & (1)_{s_3} & 0 \ldots\ldots 0 \end{pmatrix}$$

<u>is poised</u>.

The proof is similar to that of Theorem 1.6.1 and is omitted. It follows from the above method of proof that if s_2 is odd, the matrix E_n^3 is not poised.

<u>Theorem 1.6.3</u>. <u>The matrix</u>

$$\begin{pmatrix} 1 & (0)_{k-2} & (1)_{p_1} & 0 \dots 0 \\ 1 & (0)_{k-2} & (1)_{p_2} & 0 \dots 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (0)_{k-2} & (1)_{p_k} & 0 \dots 0 \end{pmatrix}$$

<u>where</u> $p_1+p_2+ \dots p_k+k = n$ <u>and</u> $p_2, \dots p_{k-1}$ <u>are even is poised</u>.

<u>Proof</u>. Consider the matrix F_{n-1}^k obtained from E_n^k by replacing the last element in the first column by zero. Then $F_{n-1}^k = G_{k-1}^k \oplus H_{n-k}^k$ where

$$G_{k-1}^k = \begin{pmatrix} 1 & \overbrace{\qquad\qquad}^{} & (0)_{k-2} \\ \cdots & \cdots & \cdots \\ 1 & \underbrace{\qquad\qquad}_{} & (0)_{k-2} \\ (0)_{k-1} & & \end{pmatrix}, \qquad H_{n-k}^k = \begin{pmatrix} (1)_{p_1} & 0 \dots 0 \\ \cdots & \cdots \\ \cdots & \cdots \\ (1)_{p_k} & 0 \dots 0 \end{pmatrix}$$

Then the polynomial $P(t) = a \prod_{j=1}^{k} (t-x_j)^{p_j}$ satisfies the homogeneous interpolation problem of H_{n-k}^k since $p_1+p_2+ \dots +p_k = n-k$.

Integrating $(k-1)$ times we have

$$R(x) = Q_{k-2}(x) + \frac{1}{(k-2)!} \int_{x_1}^{x} P(t)(x-t)^{k-2}\, dt$$

where $Q_{k-2}(x)$ is determined by the following conditions:

$$Q_{k-2}(x_k) = 0$$

$$Q_{k-2}(x_j) = -I(x_j)\ , \quad \text{where} \quad I(x) = -\frac{1}{(k-2)!} \int_{x_1}^{x} P(t)(x-t)^{k-2}\, dt\ ,$$

$$(2 \leq j \leq k-1).$$

Hence by the Lagrange interpolation formula we have

$$Q_{k-2}(x) = -\sum_{j=2}^{k-1} \ell_j(x)\, I(x_j)\ , \quad \ell_j(x) = \prod_{\substack{i=1\\ i\neq j}}^{k-1} \frac{x - x_i}{x_j - x_i}\ .$$

Now if $R(x_k) = 0$ then $R(x)$ satisfies the homogeneous interpolation problem of E_n^k. $R(x_k) = 0$ implies

$$I(x_k) - \sum_{j=2}^{k-1} \ell_j(x_k)\, I(x_j) = 0 \tag{1.6.6}$$

$$I(x_k) = \frac{1}{(k-2)!} \int_{x_1}^{x_k} P(t)(x_k - t)^{k-2}\, dt$$

$$= \frac{1}{(k-2)!} \int_{x_1}^{x_k} P(t) \sum_{j-1}^{k-1} (x_j - t)^{k-2}\, \ell_j(x_k)\ dt\ . \tag{1.6.7}$$

From (1.6.6) and (1.6.7) we have

$$\frac{1}{(k-2)!} \sum_{j=1}^{k-1} \ell_j(x_k) \sum_{p=j}^{k-1} \int_{x_p}^{x_{p+1}} P(t)(x_j - t)^{k-2}\, dt = 0\ .$$

By interchanging the order of summation we have

$$\frac{1}{(k-2)!} \sum_{p=1}^{k-1} \int_{x_p}^{x_{p+1}} P(t) \sum_{j=1}^{p} (x_j - t)^{k-2}\, \ell_j(x_k) = 0\ . \tag{1.6.8}$$

Now

$$\sum_{j=1}^{p} (x_j - t)^{k-2}\, \ell_j(x_k) = \omega(x_k) \sum_{j=1}^{p} \frac{(x_j - t)^{k-2}}{(x_k - x_j)\omega'(x_j)}\ , \quad \omega(x) = \prod_{1}^{k-1} (x - x_j)\ .$$

Denoting the divided difference of a given function f on $x_1, x_2, \ldots x_{k-1}$ by $\{x_1, x_2, \ldots x_{k-1}; f\}$, we have

$$\sum_{j=1}^{p} \frac{(x_j - t)^{k-2}}{(x_k - x_j)\omega'(x_j)} = \{x_1, x_2, \ldots x_{k-1}; f_p\}$$

where we set

$$f_p(x) = \frac{(x-t)^{k-2}}{x_k-x} , \qquad x \le x_p$$

$$= 0 , \qquad x > x_p .$$

Now it is known that

$$\{x_1, x_2, \dots x_{k-1}; f_p\} = \frac{f_p^{(k-2)}(\zeta_p)}{(k-2)!} , \quad x_1 < \zeta = \zeta_p < x_{k-1}$$

$$= \begin{cases} \dfrac{(x_k-t)^{k-2}}{(x_k-\zeta_p)^{k-1}} , & \text{if } \zeta_p \le x_p \\ 0 \text{ if } x_p < \zeta_p . \end{cases}$$

Therefore from (1.6.8) we have

$$\sum_{p=1}^{k-1} \int_{x_p}^{x_{p+1}} P(t) \{x_1, x_2, \dots x_{k-1}; f_p\}\, dt = 0 . \tag{1.6.9}$$

It is easy to see that not all the above divided differences $\{x_1, x_2, \dots x_{k-1}; f_p\}$ are zero, but each of them are non-negative. Hence (1.6.9) is valid only if a = 0. thus the homogeneous interpolation problem for E_n^k has only the trivial solution. This proves the result.

Corollary. The matrix

$$E_{3k}^k = \begin{pmatrix} 1 & (0)_{k-2} & (1)_2 & (0)_{2k-1} \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ 1 & (0)_{k-2} & (1)_2 & (0)_{2k-1} \end{pmatrix}$$

is poised.

This follows by taking $p_1 = p_2 = \ldots = p_k = 2$ in Theorem 1.6.3. It is interesting that even this special case has been overlooked in the studies of Poritsky and of Németh.

Theorems 1.6.1 and 1.6.3 can be further generalized as follows:

Theorem 1.6.4. Let $E_n^k = F_{m-1}^k \oplus G_{n-m+1}^k$, $m \leq k$ be an incidence matrix satisfying the following conditions:

(i) F_{m-1}^k has m non-zero entries only in the first column

and (ii) E_n^k is either conservative or contains at most one non-conservative even sequence in each row. Then the matrix E_n^k is poised.

The proof is similar to that of Theorem 1.6.3. Theorem 1.6.2 is not a special case of Theorem 1.6.4 since the matrix E_n^3 of Theorem 1.6.2 has two non-conservative even sequences in the second row.

Theorems 1.5.2, 1.6.2 and 1.6.4 lend support to the following conjecture:

* Conjecture: A simple incidence matrix E_n^k satisfying condition (P) is poised if and only if it has no non-conservative odd sequences.

7. Applications.

Given k real numbers $x_1 < x_2 < \ldots < x_k$ the problem of determining a polynomial $P(x)$ of degree $kn-1$ such that

$$P^{(mk)}(x_j) = y_j^{(m)}, \quad 1 \leq j \leq k, \quad m = 0, 1, \ldots, n-1$$

has been treated by Poritsky [22]. The existence and uniqueness of these polynomials is an immediate consequence of our Theorem 1.3.2 since the incidence

* This conjecture has been recently proved in one direction by Atkinson and Sharma [1].

matrix of this interpolation problem is partitioned into blocks of q-H matrices.

Ńemeth [16] has considered a more general set up than that of Poritsky. For the details of his problem we refer to §2 of Introduction. This reduces to the case of Poritsky when $p_0 = p_1 = \ldots = p_k = k$, and $x_i^{(0)} = x_i^{(1)} = \ldots = x_i^{(k-1)}$ for $1 \leq i \leq p_0$.

If $p_0 = p_1 = \ldots = p_k = 1$, and $x_1^{(0)} \neq x_1^{(1)} \neq \ldots \neq x_1^{(k-1)}$, we get the Abel-Gontcharoff interpolation. The incidence matrix of Ńemeth's general case is A-H-B and Theorem 1.3.2 applies.

8. Trigonometric Interpolation.

In the above sections we have been considering interpolation by power polynomials. For k given numbers in $[0,2\pi)$ let us find a trigonometric polynomial of order m satisfying the given conditions corresponding to an incidence matrrix $E_n^k \left(n = 2m+1 \right)$. The question then naturally arises if our classification of weakly q-H matrices applies. In other words, if E_n^k is weakly q-H or if E_n^k satisfies the conditions of Theorems 1.5.2, 1.6.1, 1.6.2 or 1.6.3 is the problem of trigonometric interpolation poised? If so, we shall say that E_n^k is T-poised.

It is easy to see that the matrix

$$E_3^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is not T-poised, since the determinant

$$\Delta = \begin{vmatrix} 1 & \sin\theta_1 & \cos\theta_1 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 1 & \sin\theta_3 & \cos\theta_3 \end{vmatrix} \begin{array}{l} = 2\sin\left(\dfrac{\theta_3-\theta_1}{2}\right)\sin\left(\dfrac{\theta_1+\theta_3}{2}-\theta_2\right) \\ = 0 \text{ if } \theta_1+\theta_3 = 2\,\theta_2 \,. \end{array}$$

It is known that

$$T_n(\theta) = x^{-n} P_{2n}(x) , \qquad x = e^{i\theta} .$$

From this it can be easily shown that if E_n^k is a Hermite matrix then it is T-poised.

However it is not at once clear whether the same result is true for weakly q-H or even A-H-B matrices. It can be shown that Abel-Gontcharoff interpolation is not poised in the trigonometric case. The above questions can be asked for other Tchebycheff systems satisfying the Wronskian condition on a given interval (a,b). But our object here is only to pose the problem and not to suggest a solution. However it can be shown that there are matrices which are A-H-B and which satisfy condition (P) but are not T-poised. For example

$$E_3^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{is an A-H-B matrix satisfying condition (P) .}$$

Since

$$\Delta = \begin{vmatrix} 1 & \sin\theta_1 & \cos\theta_1 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \cos\theta_3 & -\sin\theta_3 \end{vmatrix}$$

$$= -\sin(\theta_3 - \theta_2)$$

$$= 0 \quad \text{if} \quad \theta_3 = \pi + \theta_2 ,$$

E_3^3 is not T-poised.

CHAPTER II

(0,2) INTERPOLATION ON LEGENDRE ABSCISSAS

1. Introduction. Here we are interested in finding the interpolatory polynomials $S_n(x)$ of degree $\leq 2n-1$, where the values and the second derivatives are given at the zeros $\{x_\nu\}_{\nu=2}^{n+1}$ of $P_n(x)$, the Legendre polynomial of degree n where $x_2 > x_3 > \ldots > x_{n+1}$. We note that the problem of existence and uniqueness of these interpolatory polynomials in this case is already solved by a more general theorem of J. Suranyi and P. Turán [31]. They have proved that these polynomials exist uniquely when n is even but for n odd there is in general no polynomial of degree $\leq 2n-1$ such that for given α_ν and β_ν,

$$S_n(x_\nu) = \alpha_\nu \; ; \quad S_n''(x_\nu) = \beta_\nu \quad , \quad \nu = 2, \ldots, n+1 \; .$$

If there exists such a polynomial then there is an infinity of them.

When n is even we have the following representation for $S_n(x)$:

$$S_n(x) = \sum_{\nu=2}^{n+1} \alpha_\nu \lambda_\nu (x) + \sum_{\nu=2}^{n+1} \beta_\nu \mu_\nu (x) \tag{2.1.1}$$

where the polynomials $\lambda_\nu(x)$ and $\mu_\nu(x)$ are the fundamental polynomials of first and of second kind of this interpolation.

In [21(1)] we have considered the interpolatory polynomials $R_n(x)$ on the zeros x_ν $(\nu = 1, 2, \ldots, n+2)$ of $(1-x^2) P_n(x)$ (where $P_n(x)$ is

the n^{th} Legendre polynomial) such that

$$R_n(x_\nu) = \alpha_\nu \qquad (1 \leq \nu \leq n+2)$$

and

$$R''_n(x_\nu) = \beta_\nu \qquad (2 \leq \nu \leq n+1) .$$

Since $S_n(x)$ is a polynomial of degree $\leq$ 2n-1 and $R_n(x)$ is a polynomial of degree $\leq$ 2n+1 one can easily obtain the following relation between the fundamental polynomials of $R_n(x)$ and those of $S_n(x)$.

(2.1.2) $$\mu_\nu(x) = q_\nu\, r_1(x) + q'_\nu\, r_{n+2}(x) + \rho_\nu(x) , \qquad 2 \leq \nu \leq n+1 ,$$

and

(2.1.3) $$\lambda_\nu(x) = p_\nu\, r_1(x) + p'_\nu\, r_{n+2}(x) + r_\nu(x) , \qquad 2 \leq \nu \leq n+1 ,$$

where

(2.1.4) $$q_\nu + q'_\nu = \frac{1}{\{P'_n(x_\nu)\}^2}$$

(2.1.5) $$q_\nu - q'_\nu = \frac{A_\nu}{n(n+1)\, P'_n(x_\nu)}$$

(2.1.6) $$p_\nu + p'_\nu = \frac{2n^2 - x_\nu^2\, (2n^2 + 1)}{(2n-1)(1-x_\nu^2)^2\, [P'_n(x_\nu)]^2}$$

(2.1.7) $$p_\nu - p'_\nu = \frac{B_\nu + A_\nu\, C_\nu\, (1-x_\nu^2)}{n(n+1)(1-x_\nu^2)\, P'_n(x_\nu)}$$

(2.1.8) $$r_1(x) = \frac{1+x}{2} P_n^2(x) - \frac{(1-x^2)}{2} P_n(x)\, P'_n(x) - \frac{(1-x^2)^{1/2}}{2} P_n(x) \int_{-1}^{x} P'_n(t)(1-t^2)^{-1/2}\, dt$$

$$r_{n+2}(x) = \frac{1-x}{2} P_n^2(x) + \frac{1-x^2}{2} P_n(x)\, P_n'(x) - \frac{(1-x^2)^{1/2}\, P_n(x)}{2} \int_{-1}^{x} P_n'(t)(1-t^2)^{-1/2}\, dt, \tag{2.1.9}$$

$$\begin{aligned} r_\nu(x) &= \frac{1-x^2}{2(1-x_\nu^2)} \ell_\nu^2(x) + \frac{(1-x^2)\, P_n'(x)\, \ell_\nu(x)}{2(1-x_\nu^2)\, P_n'(x_\nu)} \\ &+ \frac{P_n(x)\, (1-x^2)^{1/2}}{2(1-x_\nu^2)\, P_n'(x_\nu)} \left[B_\nu \int_{-1}^{x} P_n(t)(1-t^2)^{-1/2}\, dt \right. \\ &\left. - \int_{-1}^{x} t\, \ell_\nu'(t)(1-t^2)^{-1/2}\, dt \right] + C_\nu\, \rho_\nu(x) , \end{aligned} \tag{2.1.10}$$

where

$$B_\nu \int_{-1}^{1} P_n(t)(1-t^2)^{-1/2}\, dt = \int_{-1}^{1} t\, \ell_\nu'(t)(1-t^2)^{-1/2}\, dt ,$$

$$C_\nu = \frac{n(n+1)}{1-x_\nu^2} - \frac{x_\nu^2}{(1-x_\nu^2)^2} , \qquad (2 \le \nu \le n+1)$$

and

$$\rho_\nu(x) = \frac{(1-x^2)^{1/2}\, P_n(x)}{2\, P_n'(x_\nu)} \left[A_\nu \int_{-1}^{x} P_n(t)(1-t^2)^{-1/2}\, dt + \int_{-1}^{x} \ell_\nu(t)(1-t^2)^{-1/2}\, dt \right] \tag{2.1.11}$$

where

$$A_\nu \int_{-1}^{1} P_n(t)(1-t^2)^{-1/2}\, dt = - \int_{-1}^{1} \ell_\nu(t)(1-t^2)^{-1/2}\, dt .$$

For the proof of (2.1.8)-(2.1.11) see [21(1)]. (2.1.4)-(2.1.7) are obtained by comparing the coefficients of x^{2n+1} and x^{2n} in (2.1.2) and (2.1.3) :

Here we shall prove the following convergence theorem concerning the polynomials $S_n(x)$.

Theorem 2.1.1. Let $f(x)$ be twice differentiable in $[-1, 1]$ and $f''(x) \varepsilon \text{ Lip } \alpha$ $(\alpha > 1/2)$ then for $\alpha_\nu = f(x_\nu)$ and $\beta_\nu = f''(x_\nu)$ the sequence of polynomials $S_n(x; f)$ converges uniformly to $f(x)$ in $[-1, 1]$.

2. We shall later need the following well-known results about Legendre polynomials (see Sansone [24]).

For $-1 \leq x \leq 1$, we have

(2.2.1) $$n^{1/2} (1-x^2)^{1/4} |P_n(x)| \leq \sqrt{2/\pi}$$

(2.2.2) $$(1-x^2)^{3/4} |P'_{n-1}(x)| \leq \sqrt{2n} ,$$

(2.2.3) $$|P_n(x)| \leq 1$$

and

(2.2.4) $$\sum_{\nu=2}^{n+1} \frac{1}{(1-x_\nu^2) [P'_n(x_\nu)]^2} = 1 .$$

From Szego [32(1)] pp.236 we have

(2.2.5) $$|P'_n(\cos \theta_\nu)| \sim \nu^{-3/2} n^2 \quad \text{for} \quad 0 < \theta_\nu \leq \pi/2 ,$$

(2.2.6) $$|P'_n(\cos \theta_\nu)| \sim (n-\nu)^{-3/2} n^2 \text{ for } \pi/2 < \theta_\nu < \pi ,$$

(2.2.7) $$(1-x_\nu^2) > \frac{\nu^2}{n^2} \text{ for } \nu = 2, 3, \ldots n/2$$

and

(2.2.8) $$(1-x_\nu^2) > \frac{(n-\nu)^2}{n^2} \quad \text{for} \quad \nu = \frac{n+1}{2}, \ldots n+1 .$$

3. In order to prove our main theorem we need the following lemmas.

Lemma 2.3.1. Let $f(x)$ be twice differentiable in $[-1, 1]$ and $f''(x) \in \text{Lip}\ \alpha$ $(0 < \alpha < 1)$ then there is a sequence $\{\Phi_n(x)\}$ of degree n at most with the following properties:

(2.3.1) $$|f(x) - \Phi_n(x)| \leq \frac{c_1}{n^{2+\alpha}} \left\{ (1-x^2)^{(2+\alpha)/2} + \frac{1}{n^{2+\alpha}} \right\}$$

and

(2.3.2) $$|f''(x) - \Phi_n''(x)| \leq \frac{c_2}{n^\alpha} \left\{ (1-x^2)^{\alpha/2} + \frac{1}{n^\alpha} \right\}$$

Formula (2.3.1) is due to Dzyadyk [6] and (2.3.2) is due to Saxena [25(2)] .

Lemma 2.3.2. For $-1 \leq x \leq 1$, we have

(2.3.3) $$\sum_{\nu=2}^{n+1} |r_\nu(x)| \leq 994\ n \log n$$

and

(2.3.4) $$\sum_{\nu=2}^{n+1} |\rho_\nu(x)| \leq \frac{105}{n} .$$

Lemma 2.3.3. For $-1 \leq x \leq 1$, we have

(2.3.5) $$\left|(1-x^2)^{1/4} \int_{-1}^{x} P_n'(t)(1-t^2)^{-1/2}\, dt\right| \leq 21\ n$$

(2.3.6) $$|C_\nu| \leq \frac{3n^2}{1-x_\nu^2}$$

(2.3.7) $$|B_\nu| \leq \frac{18\ n^{5/2}}{(1-x_\nu^2)^{7/4}[P_n'(x_\nu)]^2}$$

and

$$|A_\nu| \le \frac{12\ n}{(1-x_\nu^2)^{7/4}\ [P_n'(x_\nu)]^2} \quad . \tag{2.3.8}$$

For the proof of lemma 2.3.2 and lemma 2.3.3 see [21(1)].

<u>Lemma 2.3.4.</u> <u>For</u> $-1 \le x \le 1$, <u>we have</u>

$$\sum_{\nu=2}^{n+1} |\lambda_\nu(x)| \le 1190\ n^{3/2} \ .$$

<u>Proof</u>. From (2.1.3) and (2.1.8)-(2.1.10) we have

$$\begin{aligned}\lambda_\nu(x) &= (p_\nu + p_\nu') \left[\frac{1}{2} P_n^2(x) - \frac{(1-x^2)\ P_n(x)}{2} \int_{-1}^{x} P_n'(t)(1-t^2)^{-1/2}\, dt\right] \\ &+ (p_\nu - p_\nu') \left[\frac{x\ P_n^2(x)}{2} - \frac{(1-x^2)\ P_n(x)\ P_n'(x)}{2}\right] \\ &+ r_\nu(x)\ .\end{aligned} \tag{2.3.9}$$

Hence from (2.3.9), (2.1.6) and (2.1.7) we get

$$\begin{aligned}\lambda_\nu(x) &= \frac{2n^2 - x_\nu^2\ (2n^2+1)}{(2n-1)(1-x_\nu^2)[P_n'(x_\nu)]^2} \left[\frac{P_n^2(x)}{2} - \frac{(1-x^2)^{1/2}\ P_n(x)}{2}\right. \\ &\times \left.\int_{-1}^{x} P_n'(t)(1-t^2)^{-1/2} dt\right] + \frac{B_\nu + A_\nu C_\nu (1-x_\nu^2)}{n(n+1)(1-x_\nu^2)P_n'(x_\nu)} \left[\frac{x\ P_n^2(x)}{2} -\right. \\ &\left. - \frac{(1-x^2)\ P_n'(x)\ P_n(x)}{2}\right] \\ &+ r_\nu(x)\ .\end{aligned} \tag{2.3.10}$$

From (2.3.10) we have

$$\sum_{\nu=2}^{n+1} |\lambda_\nu(x)| \le 5n \sum_{\nu=2}^{n+1} \frac{P_n^2(x)}{2(1-x_\nu^2)[P_n'(x_\nu)]^2} \tag{2.3.11}$$

$$+ 5n \sum_{\nu=2}^{n+1} \frac{(1-x^2)^{1/4} \, |P_n(x)|}{2(1-x_\nu^2)[P_n'(x_\nu)]^2} \left| (1-x^2)^{1/4} \int_{-1}^{x} P_n'(t)(1-t^2)^{-1/2} \, dt \right|$$

$$+ \sum_{\nu=2}^{n+1} \frac{|B_\nu| + |A_\nu C_\nu| (1-x_\nu^2) \, P_n^2(x)}{2n^2(1-x_\nu^2) \, |P_n'(x_\nu)|}$$

$$+ \sum_{\nu=2}^{n+1} \frac{|B_\nu| + |A_\nu C_\nu| (1-x_\nu^2)(1-x^2) \, |P_n(x) \, P_n'(x)|}{2n^2(1-x_\nu^2) \, |P_n'(x_\nu)|}$$

$$+ \sum_{\nu=2}^{n+1} |r_\nu(x)|$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 \quad \text{(say)} .$$

Now from formulae (2.2.5)-(2.2.8) and (2.2.3) we have

(2.3.12) $$I_1 \leq 3 n .$$

From (2.2.1), (2.2.2), (2.2.5)-(2.2.8) and lemma 2.3.3, we have

(2.3.13) $$I_2 \leq 63 n^{3/2} .$$

From (2.2.5)-(2.2.8) and lemma 2.3.3 we have

(2.3.14) $$I_3 \leq 54 n^{1/2} \log n .$$

Again from (2.2.5)-(2.2.8), (2.2.1), (2.2.2) and lemma 2.3.3, we have

(2.3.15) $$I_4 \leq 76 n \log n .$$

Further from lemma 2.3.2, we have

(2.3.16) $$I_5 \leq 994 n \log n .$$

Hence from (2.3.11)-(2.3.16) the lemma follows.

Lemma 2.3.5. For $-1 \leq x \leq 1$, we get

$$\sum_{\nu=2}^{n+1} |\mu_\nu(x)| \leq 157\ n^{1/2} .$$

Proof. From (2.1.2) and (2.1.8)-(2.1.10) we have

$$\mu_\nu(x) = (q_\nu + q_\nu') \left[\frac{P_n^2(x)}{2} - \frac{(1-x^2)^{1/2} P_n(x)}{2} \int_{-1}^{x} P_n'(t)(1-t^2)^{-1/2}\, dt\right] + (q_\nu - q_\nu') \left[\frac{x}{2} P_n^2(x) - \frac{1-x^2}{2} P_n(x)\, P_n'(x_\nu)\right] + \rho_\nu(x) . \tag{2.3.17}$$

From (2.3.17) (2.1.4) and (2.1.5) we have

$$\mu_\nu(x) = \frac{1}{[P_n'(x_\nu)]^2} \left[\frac{P_n^2(x)}{2} - \frac{(1-x^2)^{1/2} P_n(x)}{2} \int_{-1}^{x} P_n'(t)(1-t^2)^{-1/2}\, dt\right] + \frac{A_\nu}{n(n+1)\, P_n'(x_\nu)} \left[\frac{x}{2} P_n^2(x) - \frac{1-x^2}{2}\, P_n(x)\, P_n'(x)\right] + \rho_\nu(x) . \tag{2.3.18}$$

Further from (2.3.18) we have

$$\begin{aligned} \sum_{\nu=2}^{n+1} |\mu_\nu(x)| \leq & \sum_{\nu=2}^{n+1} \frac{P_n^2(x)}{2[P_n'(x_\nu)]^2} + \sum_{\nu=2}^{n+1} \frac{|A_\nu|\ P_n^2(x)}{2n^2\ |P_n'(x_\nu)|} \\ & + \sum_{\nu=2}^{n+1} \frac{(1-x^2)^{1/4}\ |P_n(x)|}{2[P_n'(x_\nu)]^2} \left[(1-x^2)^{1/4} \int_{-1}^{x} P_n'(t)(1-t^2)^{-1/2}\, dt\right] \\ & + \sum_{\nu=2}^{n+1} \frac{|A_\nu|\ |(1-x^2)^{1/4} P_n(x)|\ |(1-x^2)^{1/4} P_n'(x)|}{2n^2\ |P_n'(x_\nu)|} \\ & + \sum_{\nu=2}^{n+1} |\rho_\nu(x)| \\ = & \ S_1 + S_2 + S_3 + S_4 + S_5 \quad \text{(say)} . \end{aligned} \tag{2.3.19}$$

From (2.2.3), (2.2.5) and (2.2.6) we have

(2.3.20) $$S_1 \le 1 .$$

From (2.2.1), (2.2.5), (2.2.6) and lemma 2.3.3, we have

(2.3.21) $$S_2 \le 21\, n^{1/2} .$$

From (2.2.1), (2.2.2), (2.2.5)-(2.2.8) and lemma 2.3.3, we have

(2.3.22) $$S_4 \le 18\, n^{-3/2} .$$

From (2.2.3), (2.2.5), (2.2.6) and lemma 2.3.3, we have

(2.3.23) $$S_3 \le 12\, n^{-3/2} .$$

Further from lemma 2.3.2, we have

(2.3.24) $$S_5 \le \frac{105}{n} .$$

Hence from (2.3.19)-(2.3.24) we get the required result.

4. Proof of the Theorem 2.1.1. Due to the uniqueness theorem we have

(2.4.1) $$\phi_n(x) = \sum_{\nu=2}^{n+1} \phi_n(x_\nu)\, \lambda_\nu(x) + \sum_{\nu=2}^{n+1} \phi''_n(x_\nu)\, \mu_\nu(x)$$

where $\phi_n(x)$ is defined by the lemma 2.3.1 and by (2.1.1)

(2.4.2) $$S_n(x;f) = \sum_{\nu=2}^{n+1} f(x_\nu)\, \lambda_\nu(x) + \sum_{\nu=2}^{n+1} f''(x_\nu)\, \mu_\nu(x) .$$

Since

(2.4.3) $$|S_n(x;f) - f(x)| \le |S_n(x;f) - \phi_n(x)| + |\phi_n(x) - f(x)|$$

then using (2.4.1) and (2.4.2), we have

(2.4.4)
$$|S_n(x;f) - \phi_n(x)| \le \sum_{\nu=2}^{n+1} |f(x_\nu) - \phi_n(x_\nu)|\ |\lambda_\nu(x)|$$

$$+ \sum_{\nu=2}^{n+2} |f''(x_\nu) - \phi_n''(x_\nu)|\ |\mu_\nu(x)|$$

$$= U_1 + U_2 \quad \text{(say)} .$$

From lemma 2.3.1 and lemma 2.3.4,we have

(2.4.5)
$$U_1 \le \frac{1190\ C_1\ n^{3/2}}{n^{2+\alpha}}$$

$$= o(1) .$$

From lemma 2.3.1 and lemma 2.3.5,we have

(2.4.6)
$$U_2 \le \frac{157\ n^{1/2}\ C_2}{n^{\alpha}}$$

$$= o(1) .$$

Hence from (2.4.5), (2.4.6) and lemma 2.3.1 we have for $-1 \le x \le 1$

$$|S_n(x;f) - f(x)| = o(1) .$$

This completes the proof of the theorem.

CHAPTER III

(0,2) INTERPOLATION ON JACOBI ABSCISSAS

1. In the literature on (0,2) interpolation (See [3] and [2]) all the authors have considered the case when the nodes are symmetrical. However Prasad and Saxena [20] investigated the interpolatory polynomials taking the zeros of $x\,L_n'(x)$ as nodes where $L_n(x)$ is the Laguerre polynomial of degree $\leq n$. Except in [20] and (see below [35]), the problem which involves the investigation of interpolatory polynomials when the nodes are unsymmetrical (e.g., when the nodes are the zeros of $P_n^{(\alpha,-\alpha)}(x)$, the Jacobi polynomial of degree n) seems not to have been treated. A. Sharma conjectured in this case the same type of general theorem as that by P. Turán and his associates [31] in the symmetrical case. The proof of this conjecture given below fills to some extent the gap in the literature on (0,2) interpolation on Jacobi abscissas.

The object of this chapter is to discuss the problems of existence, uniqueness for polynomials $R_n(x)$ of degree $\leq 2n+1$ for (0,2) interpolation when the nodes are the zeros of $\omega_{n,\alpha}(x) \equiv \omega_n(x) \equiv P_n^{(\alpha,-\alpha)}(x)$. The uniform convergence has been proved only for $\alpha = 1/3$. Two particular cases have been previously investigated: the case $\alpha = 0$ and n even by Prasad and Varma [21(1)], and the case $\alpha = \frac{1}{2}$ and $n \geq 1$ by Varma and Gupta [35].

2. Let

$$-1 = x_{n+1} < x_n < \dots < x_1 < x_0 = 1 \tag{3.2.1}$$

be the zeros of $(1-x)^2\,\omega_n(x)$. Here we are interested in determining the

interpolatory polynomials $R_n(x)$ of degree $\leq 2n+1$ satisfying the following conditions:

$$(3.2.2) \qquad \begin{aligned} R_n(x_\nu) &= \alpha_\nu \; ; \qquad && \nu = 0,1,2, \dots n+1 \; , \\ R''_n(x_\nu) &= \beta_\nu \; ; && \nu = 1,2,3, \dots n \quad . \end{aligned}$$

It turns out that in this case the polynomials are unique for both n even and for n odd. Obviously, $R_n(x)$ is given by

$$(3.2.3) \qquad R_n(x) = \sum_{\nu=0}^{n+1} \alpha_\nu \, r_\nu(x) + \sum_{\nu=1}^{n} \beta_\nu \, \rho_\nu(x) \quad ,$$

where $r_\nu(x)$ and $\rho_\nu(x)$ are given by Theorem 3.2.2, satisfying the following conditions:

$$r_\nu(x_j) = \begin{cases} 0, & j \neq \nu \\ 1, & j = \nu \end{cases} , \qquad j, \nu = 0,1,2, \dots n+1$$

$$r''_\nu(x_j) = 0 \quad , \qquad 1 \leq j \leq n \; , \; 0 \leq \nu \leq n+1$$

and

$$\rho_\nu(x_j) = 0 \quad , \qquad 0 \leq j \leq n+1 \; , \; 1 \leq \nu \leq n$$

$$\rho''_\nu(x_j) = \begin{cases} 1, & j = \nu \\ 0, & j \neq \nu \end{cases} , \qquad 1 \leq j \leq n \; , \; 1 \leq \nu \leq n \; .$$

We shall prove the following theorems:

<u>Theorem 3.2.1.</u> <u>If</u> x_ν $(0 \leq \nu \leq n+1)$ <u>are given by</u> (3.2.1) , $0 \leq \alpha < 1$ <u>and</u> α_ν , β_ν <u>are any arbitrary given values then there exists one and only one polynomial</u> $g_{2n+1}(x)$ <u>of degree</u> $\leq 2n+1$ <u>which satisfies</u> (3.2.2).

Theorem 3.2.2. For $\nu = 1,2, \ldots n$ and $0 < \alpha < 1$, we have

$$(3.2.4) \qquad \rho_\nu(x) = \frac{(1-x^2)^{1/2}\omega_n(x)(1-x)^{\alpha/2}}{2\,\omega_n'(x_\nu)(1+x)^{\alpha/2}}\left[A_\nu \int_{-1}^{x} p(\alpha,t)\,\omega_n(t)\,dt + \int_{-1}^{x} p(\alpha,t)\,\ell_\nu(t)\,dt\right]$$

where

$$(3.2.5) \qquad \ell_\nu(x) = \frac{\omega_n(x)}{(x-x_\nu)\,\omega_n'(x_\nu)},$$

$$(3.2.6) \qquad p(\alpha,t) = (1+t)^{(\alpha-1)/2}\,(1-t)^{-(1+\alpha)/2}$$

$$(3.2.7) \qquad A_\nu \int_{-1}^{1} p(\alpha,t)\,\omega_n(t)\,dt = -\int_{-1}^{1} p(\alpha,t)\,\ell_\nu(t)\,dt\,.$$

For $1 \leq \nu \leq n$ we have

$$(3.2.8) \qquad r_\nu(x) = \frac{(1-x^2)}{(1-x_\nu^2)}\,\ell_\nu^2(x) + \frac{(1-x^2)\omega_n(x)\ell_\nu'(x)}{2(1-x_\nu^2)\,\omega_n'(x_\nu)} + \frac{(1-x^2)^{1/2}(1-x)^{\alpha/2}\omega_n(x)}{2(1-x_\nu^2)(1+x)^{\alpha/2}\omega_n'(x_\nu)}\left[B_\nu \int_{-1}^{x} p(\alpha,t)\,\omega_n(t)\,dt + \int_{-1}^{x} (3\alpha+t)\,p(\alpha,t)\,\ell_\nu'(t)\,dt\right] + C_\nu\,\rho_\nu(x)$$

where

$$(3.2.9) \qquad B_\nu \int_{-1}^{1} p(\alpha,t)\,\omega_n(t)\,dt = \int_{-1}^{1} (3\alpha+t)\,\ell_\nu'(t)\,dt$$

and

(3.2.10) $$C_\nu = \frac{n(n+1)}{1-x_\nu^2} - \frac{(\alpha+x_\nu)(x_\nu+7\alpha)}{(1-x_\nu^2)^2} .$$

We also have

(3.2.11) $$r_0(x) = \frac{(1+x)\,\omega_n^2(x)}{2\left[\binom{n+\alpha}{n}\right]^2} - \frac{(1-x^2)\omega_n(x)\omega_n'(x)}{2\left[\binom{n+\alpha}{n}\right]^2}$$
$$+ \frac{(3\alpha-1)(1-x^2)^{1/2}\omega_n(x)(1-x)^{\alpha/2}}{2\left[\binom{n+\alpha}{n}\right]^2(1+x)^{\alpha/2}}\left[D\int_{-1}^{x} p(\alpha,t)\,\omega_n(t)\,dt\right.$$
$$\left.+\int_{-1}^{x} p(\alpha,t)\,\omega_n'(t)\,dt\right]$$

where

(3.2.12) $$D\int_{-1}^{1} p(\alpha,t)\,\omega_n(t)\,dt = -\int_{-1}^{1} p(\alpha,t)\,\omega_n'(t)\,dt$$

and

(3.2.13) $$r_{n+1}(x) = \frac{(1-x)\omega_n^2(x)}{2\left[\binom{n+\alpha}{n}\right]^2} + \frac{(1-x^2)\omega_n(x)\omega_n'(x)}{2\left[\binom{n+\alpha}{n}\right]^2}$$
$$+ \frac{(1+3\alpha)(1-x^2)^{1/2}(1-x)^{\alpha/2}}{2(1+x)^{\alpha/2}\left[\binom{n+\alpha}{n}\right]^2}\left[E\int_{-1}^{x} p(\alpha,t)\,\omega_n(t)\,dt\right.$$
$$\left.-\int_{-1}^{x} p(\alpha,t)\,\omega_n'(t)\,dt\right]$$

and

$$E \int_{-1}^{1} p(\alpha, t)\, \omega_n(t)\, dt = - \int_{-1}^{1} p(\alpha, t)\, \omega_n'(t)\, dt\ . \tag{3.2.14}$$

Remark. For $\alpha = 0$ and n even, the theorem is still true.

Theorem 3.2.3. Let $\alpha = \frac{1}{3}$ and $f(x)$ be twice differentiable in $[-1, 1]$. Let $f''(x) \in \text{Lip}\ \mu$ $(\mu > \frac{1}{6})$, $\alpha_\nu = f(x_\nu)$ and $\beta_\nu = f''(x_\nu)$. Then the sequence of polynomials $R_n(x; f)$ converges uniformly to $f(x)$ for $-1 + \varepsilon \leq x \leq 1 - \varepsilon$, $0 < \varepsilon < 1$ (ε being arbitrary fixed positive number).

3. We shall need the following known results for the proof of theorems of this chapter and Chapter IV.

$$(1-x_\nu^2) > \frac{\nu^2}{n^2}\ ; \quad \nu = 1, 2, \ldots, \frac{n}{2}\ , \tag{3.3.1}$$

$$(1-x_\nu^2) > \frac{(n-\nu)^2}{n^2}\ ; \quad \nu = \frac{n+1}{2}, \ldots n\ ,$$

$$(1-x^2)\omega_n''(x) + (-2\alpha-2x)\omega_n'(x) + n(n+1)\omega_n(x) = 0\ , \tag{3.3.2}$$

$$(1-x^2)\omega_n'(x) = [(n+1)x+\alpha]\omega_n(x) - (n+1)\omega_{n+1}(x)\ , \tag{3.3.3}$$

$$\int_{-1}^{1} \omega_n(x)\omega_m(x)(1+x)^{-\alpha}(1-x)^{\alpha}\, dx = 0, \quad \text{if} \quad m \neq n \tag{3.3.4}$$

$$= \frac{2\Gamma(n+\alpha+1)\ \Gamma(n-\alpha+1)}{(2n+1)n!\ \Gamma(n+1)}\ , \text{ if } \quad m = n\ .$$

(3.3.5) $\qquad \dfrac{\Gamma(n+1+\beta+\gamma)}{\Gamma(n+1+\beta)} < d\ n^{\gamma}$ (where $\gamma > -1$, $\beta > -1$, see Natanson [15] pp. 324).

$$|\omega_n(x)| = 0(n^{-1/2}) \text{ for } -1+\varepsilon \leq x \leq 1-\varepsilon\ , \tag{3.3.6}$$

$$|\omega_n'(\cos\theta_\nu)| \sim \nu^{-3/2-\alpha}\, n^{\alpha+2} \quad \text{for} \quad 0 < \theta_\nu \leq \pi/2\ , \tag{3.3.7}$$

(3.3.8) $$|\omega_n'(\cos\theta_\nu)| \sim (n-\nu)^{-3/2+\alpha} n^{-\alpha+2} \quad \text{for} \quad \pi/2 < \theta_\nu < \pi ,$$

and

(3.3.9) $$|\omega_j(x_\nu)| = \frac{0(j^{-1/2})}{(1-x_\nu^2)^{(1+2\alpha)/4}} , \quad (1 \le j \le n-1) \text{ (see Szegö [32(2)]).}$$

For (3.3.1), (3.3.2), (3.3.3), (3.3.4), (3.3.6), (3.3.7) and (3.3.8) see Szegö [32(1)] pp. 67, 68, 236, 60, 72 and 194.

(3.3.10) $$|\ell_\nu(x)| = \frac{0(n)}{(1-x_\nu^2)^{(2\alpha+5)/4}[\omega_n'(x_\nu)]^2} , \quad (-1+\varepsilon \le x \le 1-\varepsilon) ,$$

(3.3.11) $$(1-x^2)|\ell_\nu'(x)| = \frac{0(n^2)}{(1-x_\nu^2)^{(2\alpha+5)/4}[\omega_n'(x_\nu)]^2} ,$$

(3.3.12) $$\omega_n(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1)_{n+k} (-1)^k (1-t)^k}{k!(n-k)!(1+\alpha)_k (1)_n 2^k} , \text{ (Rainville [23] pp. 255),}$$

(3.3.13) $$\ell_\nu(x) = \frac{\Gamma(n+\alpha+1)\ \Gamma(n-\alpha+1)}{\{\Gamma(n+1)\}^2(1-x_\nu^2)[\omega_n'(x_\nu)]^2} \sum_{j=0}^{n-1} \frac{(2j+1)\{\Gamma(j+1)\}^2\omega_j(x)\omega_j(x_\nu)}{\Gamma(j+\alpha+1)\ \Gamma(j-\alpha+1)} ,$$

and

(3.3.14) $$\omega_n(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n , & 1+n ; & \frac{1-x}{2} \\ & 1+\alpha ; & \end{matrix}\right], \text{(Rainville [23] pp. 254).}$$

For (3.3.10), (3.3.11) and (3.3.12) see Szegö [32(1)] pp. 71.

4. Proof of the Theorem 3.2.1. Theorem 3.2.1 is proved if we could show that the polynomial $g_{2n+1}(x)$ of degree $\le 2n+1$ satisfying the conditions

(3.4.1) $$g_{2n+1}(x_\nu) = 0 \quad ; \qquad (0 \le \nu \le n+1)$$

and

(3.4.2) $$g''_{2n+1}(x_\nu) = 0 \quad ; \qquad (1 \leq \nu \leq n) ,$$

is identically zero. To prove this we write

(3.4.3) $$g_{2n+1}(x) = (1-x^2)\omega_n(x)q_{n-1}(x)$$

which satisfies the condition (3.4.1) with arbitrary polynomial $q_{n-1}(x)$. In order that the condition (3.4.2) is also satisfied we have on differentiating (3.4.3) twice and using (3.3.2)

(3.4.4) $$(1-x_j^2)q'_{n-1}(x_j) + (\alpha-x_j)q_{n-1}(x_j) = 0 \;, \; (1 \leq j \leq n)$$

whence

(3.4.5) $$(1-x^2)q'_{n-1}(x) + (\alpha-x)q_{n-1}(x) = c\;\omega_n(x)$$ where c is a numerical constant.

Setting

(3.4.6) $$q_{n-1}(x) = \sum_{k=0}^{n-1} a_k\,\omega_k(x)$$

using the recurrence relation (3.3.3) and (4.5.1) Szegö [32(1)] pp.71 and simplifying we have

(3.4.7) $$\sum_{k=1}^{n-2}\left[\frac{-k}{2k-1}\,a_{k-1} + 2\alpha\,a_k + \frac{(k+1)^2-\alpha^2}{(2k+3)}\,a_{k+1}\right]\omega_k(x)$$

$$+\left[2\alpha\,a_0 + \frac{1-\alpha^2}{3}\,a_1\right]\omega_0(x)$$

$$+ \left[2\alpha\ a_{n-1} - \frac{(n-1)^2}{(2n-3)}\ a_{n-2} \right] \omega_{n-1}(x)$$

$$- \frac{n^2}{2n-1}\ a_{n-1}\, \omega_n(x) = c\, \omega_n(x)\ .$$

From (3.4.7) we have

$$(3.4.8) \qquad \begin{cases} 2\alpha\ a_0 + \dfrac{1-\alpha^2}{3}\ a_1 = 0 \\[2ex] -\dfrac{k^2}{2k-1}\, a_{k-1} + 2\alpha\ a_k + \dfrac{(k+1)^2 - \alpha^2}{(2k+3)}\ a_{k+1} = 0 \\[1ex] \qquad\qquad\qquad\qquad\qquad\qquad (1 \leq k \leq n-2)\ , \\[2ex] -\dfrac{(n-1)^2}{(2n-3)}\ a_{n-2} + 2\alpha\ a_{n-1} = 0 \end{cases}$$

and

$$(3.4.9) \qquad - \frac{n^2}{2n-1}\ a_{n-1} = c\ .$$

From (3.4. 8) we have

$$D_n = \begin{vmatrix} 2\alpha & \frac{1-\alpha^2}{3} & 0 & 0 \ldots & 0 & 0 & 0 & 0 \\ -1 & 2\alpha & \frac{2^2-\alpha^2}{5} & 0 \ldots & 0 & 0 & 0 & 0 \\ 0 & -\frac{2^2}{3} & 2\alpha & \frac{3^2-\alpha^2}{7} \ldots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \ldots & \frac{-(n-3)^2}{(2n-7)} & 2\alpha & \frac{(n-2)^2-\alpha^2}{(2n-3)} & 0 \\ 0 & 0 & 0 & 0 \ldots & 0 & \frac{-(n-2)^2}{(2n-5)} & 2\alpha & \frac{(n-1)^2-\alpha^2}{(2n-1)} \\ 0 & 0 & 0 & 0 \ldots & 0 & 0 & \frac{-(n-1)^2}{(2n-1)} & 2\alpha \end{vmatrix}\ .$$

Hence we have

$$D_n = 2\alpha\, D_{n-1} + \frac{(n-1)^2[(n-1)^2-\alpha^2]}{(2n-1)(2n-3)}\, D_{n-2} \quad .$$

Now for $0 \le \alpha < 1$, $D_1 \ne 0$ and $D_2 \ne 0$ which can be easily seen. Therefore $D_n \ne 0$ can be shown easily by induction. So finally we have $a_0 = a_1 = \ldots = a_{n-1} = 0$. Therefore from (3.4.9) $c = 0$ and hence $g_{2n+1}(x) \equiv 0$. This completes the proof of Theorem 3.2.1. The proof of Theorem 3.2.2 can be given on the same lines as in [35].

5. Estimation of the fundamental polynomials of second kind.

Lemma 3.5.1. *We have for* $\alpha = 1/3$,

$$C_1\, n^{-1/3} < \left| \int_{-1}^{1} p(\alpha,t)\, \omega_{n;\alpha}(t)\, dt \right| < C_2\, n^{-1/3} \quad ,$$

where $p(\alpha,t)$ *is given by* (3.2.6) *and* C_1, C_2 *are positive constants*.

Proof. From (3.3.12) we have

$$\int_{-1}^{1} p(\alpha,t)\, \omega_n(t)\, dt = \int_{-1}^{1} p(\alpha,t) \sum_{k=0}^{n} \frac{(1)_{n+k}(1+\alpha)_n(-1)^k}{k!\,(n-k)!\,(1+\alpha)_k(1)_n} \left(\frac{1-t}{2}\right)^k dt$$

$$= \sum_{k=0}^{n} \frac{(1+\alpha)_n(1)_{n+k}(-1)^k\, 2^{-k}}{k!\,(n-k)!\,(1+\alpha)_k\,(1)_n} \int_{-1}^{1} (1+t)^{(\alpha-1)/2}(1-t)^{(2k-\alpha-1)/2}\, dt$$

$$= \sum_{k=0}^{n} \frac{(1+\alpha)_n(1)_{n+k}(-1)^k}{k!\,(n-k)!\,(1+\alpha)_k(1)_n}\, \beta\left(\frac{\alpha+1}{2}, \frac{2k-\alpha+1}{2}\right)$$

$$= \frac{(1+\alpha)_n\, \Gamma(\frac{1+\alpha}{2})}{(1)_n} \sum_{k=0}^{n} \frac{\Gamma(n+k+1)\,\Gamma(1+\alpha)\,\Gamma(k-\frac{\alpha}{2}+\frac{1}{2})(-1)^k}{k!\,\Gamma(n-k+1)\,\Gamma(k+\alpha+1)\,\Gamma(k+1)}$$

$$= \frac{(1+\alpha)_n\, \Gamma(\frac{1+\alpha}{2})\, \Gamma(\frac{1-\alpha}{2})}{\Gamma(n+1)}\; {}_3F_2\left[\begin{matrix} -n, & 1+n, & \frac{1-\alpha}{2}\,; \\ & 1+\alpha, & 1\,; \end{matrix}\; 1\right]$$

Making use of the Theorem 30, pp. 87, Rainville [23] we have for $\alpha = 1/3$,

$$= \frac{\Gamma(\frac{1}{3})\ \Gamma(n+\frac{2}{3})}{\Gamma(n+1)} \quad .$$

From which the lemma follows on using (3.3.5).

Remark. The upper and lower estimates for $\left|\int_{-1}^{1} p(\alpha,t)\ \omega_{n,\alpha}(t)\ dt\right|$ when $\alpha = 0$ and n even can also be obtained by the method used above. In this case we have by the above method

$$C_1^* \, n^{-1/2} < \left| \int_{-1}^{1} p(o,t)\ \omega_{n,o}(t)\ dt \right| < C_2^* \, n^{-1/2} \quad .$$

However by direct computation we have obtained the following improved estimates [21(1)]:

$$\frac{2}{n+1} < \int_{-1}^{1} p(o,t)\ \omega_{n,o}(t)\ dt < \frac{2}{n} \quad .$$

Lemma 3.5.2. We have

$$\left|\int_{-1}^{1} p(\alpha,t)\ \omega_n(t)\ dt\right| \le C_3\, n^{-1/2} \ , \quad \text{if } 0 \le \alpha < 1/4$$

$$\le C_4\, n^{-1+2\alpha} \ , \quad \text{if } 1/4 \le \alpha < 1 \ .$$

Proof. Substituting $t = \cos\theta$ one gets

$$\text{(3.5.1)} \qquad \left|\int_{-1}^{1} p(\alpha,t)\ \omega_n(t)\ dt\right| = \left|\int_0^{\pi} \frac{(1+\cos\theta)^{\alpha}\omega_n(\cos\theta)}{(\sin\theta)^{\alpha}}\ d\theta\right|$$

$$\le \int_0^{\pi} \frac{(1+\cos\theta)^{\alpha}\,|\omega_n(\cos\theta)|}{(\sin\theta)^{\alpha}}\ d\theta \ .$$

Now consider

$$(3.5.2) \qquad \int_0^{\pi} \frac{(1+\cos\theta)^{\alpha}|\omega_n(\cos\theta)|}{(\sin\theta)^{\alpha}}\, d\theta = \int_0^{\pi/2n} + \int_{\pi/2n}^{\pi-\pi/2n} + \int_{\pi-\pi/2n}^{\pi}$$

$$= I_1 + I_2 + I_3 \quad .$$

Making use of formula (7.32.2) of Szegö [32(1)] pp. 166 one obtains

$$I_1 = \int_0^{\pi/2n} \frac{(1+\cos\theta)^{\alpha}|\omega_n(\cos\theta)|}{(\sin\theta)^{\alpha}}\, d\theta$$

$$\leq \max_{0 \leq \theta \leq \pi} |\omega_n(\cos\theta)| \int_0^{\pi/2n} \theta^{-\alpha}\, d\theta$$

$$(3.5.3) \qquad \leq C_5\, n^{-1+2\alpha} \quad .$$

Similarly one can see that

$$(3.5.4) \qquad I_3 \leq C_6\, n^{-1+2\alpha} \quad .$$

Applying Theorem 8.21.13 of Szegö [32(1)] pp. 195 we have

$$(3.5.5) \qquad I_2 \leq C_7\, n^{-1/2} \int_{\pi/2n}^{\pi-\pi/2n} \frac{(1+\cos\theta)^{\alpha}|k(\theta)|}{(\sin\theta)^{\alpha}}\, d\theta$$

$$+ \; C_8\, n^{-3/2} \int_{\pi/2n}^{\pi-\pi/2n} \frac{(1+\cos\theta)^{\alpha}|k(\theta)|}{(\sin\theta)^{1+\alpha}}\, d\theta$$

$$= I_1^* + I_2^* \quad ,$$

where $k(\theta) = \pi^{-1/2} (\sin\theta/2)^{-\alpha-1/2} (\cos\theta/2)^{\alpha-1/2}$.

Now

$$(3.5.6) \qquad I_1^* = C_7\, n^{-1/2} \int_{\pi/2n}^{\pi-\pi/2n} \frac{(1+\cos\theta)^{\alpha}(\cos\theta/2)^{2\alpha}}{(\sin\theta)^{2\alpha+1/2}}\, d\theta$$

$$\leq C_8\, n^{-1/2} \int_{\pi/2n}^{\pi/2} \theta^{-(1+4\alpha)/2}\, d\theta \quad .$$

From (3.5.5) we have

$$I_1^* \le C_9 \, n^{-1/2} \;, \quad \text{if} \quad 0 \le \alpha < 1/4$$

(3.5.7) $$C_{10} \, n^{-1+2\alpha} \;, \quad \text{if} \quad 1/4 \le \alpha \le 1 \;.$$

Since

$$I_2^* \le C_{10} \, n^{-3/2} \left[\int_{\pi/2n}^{\pi/2} \frac{d\theta}{(\sin \theta)^{3/2+2\alpha}} + \int_{\pi/2}^{\pi-\pi/2n} \frac{d\theta}{(\sin \theta)^{3/2+2\alpha}} \right]$$

therefore using the inequalities $2\theta/\pi \le \sin \theta \le \theta$ for $0 \le \theta \le \pi/2$ and $\sin \theta \ge 2/\pi \, (\pi-\theta)$ for $\pi/2 \le \theta \le \pi$, we have

(3.5.8) $$I_2^* \quad C_{11} \, n^{-1+2\alpha} \quad .$$

From (3.5.5), (3.5.7) and (3.5.6) we have

(3.5.9) $$I_2 \le C_{12} \, n^{-1/2} \;, \quad \text{if} \quad 0 \le \alpha < 1/4$$

$$\le C_{13} \, n^{-1+2\alpha}, \quad \text{if} \quad 1/4 \le \alpha < 1 \;.$$

Hence from (3.5.1), (3.5.2), (3.5.3), (3.5.4) and (3.5.9) we have the required result.

Lemma 3.5.3. We have

$$\left| \int_{-1}^{1} p(\alpha,t) \, \ell_\nu(t) \, dt \right| \le \frac{C_{14} \, n}{(1-x_\nu^2)^{(5+2\alpha)/4} [\omega_n'(x_\nu)]^2} \;, \quad \text{if } 0 \le \alpha < 1/4$$

$$\le \frac{C_{15} \, n^{1/2+2\alpha}}{(1-x_\nu^2)^{(5+2\alpha)/4} [\omega_n'(x_\nu)]^2} \;, \quad \text{if } 1/4 \le \alpha < 1 \;.$$

Proof. From (3.3.13) we have

$$\left|\int_{-1}^{1} p(\alpha,t)\ell_\nu(t)\,dt\right| \le |\Delta_{n,\nu}| \sum_{j=0}^{n-1} \frac{\{\Gamma(j+1)\}^2(2j+1)|\omega_j(x_\nu)|}{\Gamma(j+1+\alpha)\,\Gamma(j-\alpha+1)} \left|\int_{-1}^{1} p(\alpha,t)\omega_j(t)\,dt\right|$$

where

$$(3.5.10)\qquad \Delta_{n,\nu} = \frac{\Gamma(n+\alpha+1)\,\Gamma(n-\alpha+1)}{\{\Gamma(n+1)\}^2\,(1-x_\nu^2)\,[\omega_n'(x_\nu)]^2} \quad .$$

From (3.3.9), (3.3.5) and lemma 3.5.2 one can at once see the required result.

Lemma 3.5.4. For $-1 \le x \le 1$, we have

$$\left|\int_{-1}^{x} p(\alpha,t)\,\omega_n(t)\,dt\right| \le C_{16}\, n^{-1/2} , \quad \text{if} \quad 0 \le \alpha < 1/4$$

$$\le C_{17}\, n^{-1+2\alpha}, \quad \text{if} \quad 1/4 \le \alpha < 1 .$$

Proof. We know that

$$\left|\int_{-1}^{x} p(\alpha,t)\;\omega_n(t)\,dt\right| \le \int_{-1}^{x} |p(\alpha,t)\;\omega_n(t)|\,dt$$

$$\le \int_{-1}^{x} p(\alpha,t)|\omega_n(t)|\,dt \quad .$$

Now substituting $t = \cos\theta$ we have

$$(3.5.11)\qquad \int_{-1}^{1} p(\alpha,t)\,|\omega_n(t)|\,dt = \int_0^{\pi} \frac{(1+\cos\theta)^\alpha|\omega_n(\cos\theta)|}{(\sin\theta)^\alpha}\,d\theta \ .$$

So from (3.5.2), (3.5.3), (3.5.4), (3.5.11) and (3.5.9) the lemma follows.

Lemma 3.5.5. For $-1 \le x \le 1$, we have

$$\left|\int_{-1}^{x} p(\alpha,t)\ell_\nu(t)\,dt\right| \le \frac{C_{18}\, n}{(1-x_\nu^2)^{(5+2\alpha)/4}[\omega_n'(x_\nu)]^2} , \quad \text{if} \quad 0 \le \alpha < 1/4$$

$$\le \frac{C_{19}\, n^{1/2+2\alpha}}{(1-x_\nu^2)^{(5+2\alpha)/4}[\omega_n'(x_\nu)]^2} , \quad \text{if} \quad 1/4 \le \alpha < 1 .$$

Proof. From (3.3.13) we have

$$\left|\int_{-1}^{x} p(\alpha,t)\ell_\nu(t)\,dt\right| \le |\Delta_{n,\nu}| \sum_{j=0}^{n-1} \frac{(2j+1)\{\Gamma(j+1)\}^2|\omega_j(x_\nu)|}{\Gamma(j+\alpha+1)\,\Gamma(j-\alpha+1)} \left|\int_{-1}^{x} p(\alpha,t)\omega_j(t)\,dt\right|$$

where $\Delta_{n,\nu}$ is given by (3.5.10) .

Now making use of lemma 3.5.4 ,(3.3.9) and (3.3.5) we have the lemma.

Lemma 3.5.6. For $-1+\varepsilon \le x \le 1-\varepsilon$ and $\alpha = 1/3$ we have

$$\sum_{\nu=1}^{n} |\rho_\nu(x)| \le C_{20}\, n^{1/6} .$$

Proof. Since from (3.2.4) ,

$$\rho_\nu(x) = \frac{(1-x^2)^{1/2}\omega_n(x)(1-x)^{\alpha/2}}{2\,\omega_n'(x_\nu)(1+x)^{\alpha/2}} \left[A_\nu \int_{-1}^{x} p(\alpha,t)\,\omega_n(t)\,dt + \int_{-1}^{x} p(\alpha,t)\,\ell_\nu(t)\,dt \right] \tag{3.5.12}$$

therefore

$$\sum_{\nu=1}^{n} |\rho_\nu(x)| \le \frac{(1-x^2)^{1/2}|\omega_n(x)|\,(1-x)^{\alpha/2}}{2\,(1+x)^{\alpha/2}} \sum_{\nu=1}^{n} \frac{1}{|\omega_n'(x_\nu)|} \times \left[|A_\nu| \left|\int_{-1}^{x} p(\alpha,t)\,\omega_n(t)\,dt\right| + \left|\int_{-1}^{x} p(\alpha,t)\ell_\nu(t)\,dt\right| \right] .$$

Applying lemma 3.5.1, lemma 3.5.3, lemma 3.5.4, lemma 3.5.5 for $\alpha = 1/3$ and using relations (3.3.6), (3.3.7),(3.3.8) and (3.3.1) we have

$$\sum_{\nu=1}^{n} |\rho_\nu(x)| \le C_{20}\, n^{1/6} .$$

This completes the proof of the lemma.

6. <u>Estimation of the fundamental polynomials of the first kind.</u>

<u>Lemma 3.6.1</u>. <u>We have for</u> $0 \leq \alpha < 1$

$$\left| \int_{-1}^{1} p(\alpha,t)\ \omega_n'(t)\ dt \right| \leq C_{22}\ n^{1+2\alpha} \quad .$$

<u>Proof</u>. Since $\omega_n'(t) = \frac{n+1}{2} P_{n-1}^{(1+\alpha,1-\alpha)}(t)$ (See Rainville [23] formula (2) pp. 263), we have

$$\int_{-1}^{1} p(\alpha,t)\ \omega_n'(t)\ dt = \frac{(n+1)}{2} \int_{-1}^{1} p(\alpha,t)\ P_{n-1}^{(1+\alpha,1-\alpha)}(t)\ dt \quad . \tag{3.6.1}$$

On substituting $t = \cos\theta$ in (3.6.1) we have

$$\frac{(n+1)}{2} \int_{-1}^{1} p(\alpha,t)\ P_{n-1}^{(1+\alpha,1-\alpha)}(t)\ dt = \frac{(n+1)}{2} \int_{0}^{\pi} \frac{(1+\cos\theta)^{\alpha} P_{n-1}^{(1+\ ,1-\ )}(\cos\theta)}{(\sin\theta)^{\alpha}}\ d\theta$$

Now applying the same technique as in the proof of lemma 3.5.2 we have

$$\left| \int_{0}^{\pi} \frac{(1+\cos\theta)^{\alpha}\ P_{n}^{(1+\alpha,1-\alpha)}(\cos\theta)}{(\sin\theta)^{\alpha}}\ d\theta \right| \leq C_{21}\ n^{2\alpha} \quad . \tag{3.6.2}$$

Hence from (3.6.1) and (3.6.2) the lemma follows.

<u>Lemma 3.6.2</u>. <u>We have for</u> $0 \leq \alpha < 1$,

$$\left| \int_{-1}^{1} (3\alpha+t)\ p(\alpha,t)\ell_{\nu}'(t)\ dt \right| \leq \frac{C_{23}\ n^{5/2+2\alpha}}{(1-x_{\nu}^2)^{(5+2\alpha)/4}[\omega_n'(x_{\nu})]^2}$$

<u>Proof</u>. With the help of (3.3.13) we have

$$\left| \int_{-1}^{1} (3\alpha+t)\ p(\alpha,t)\ell_{\nu}'(t)\ dt \right|$$

$$\leq |\Delta_{n,\nu}| \sum_{j=0}^{n-1} \frac{(2j+1)\{\Gamma(j+1)\}^2 |\omega_j(x_{\nu})|}{\Gamma(j+\alpha+1)\ \Gamma(j-\alpha+1)} \left| \int_{-1}^{1} (3\alpha+t)p(\alpha,t)\omega_j'(t)\ dt \right| \ .$$

Applying the same method as in the proof of lemma 3.6.1, we have the lemma.

<u>Lemma 3.6.3.</u> <u>For</u> $-1 \le x \le 1$, <u>we have for</u> $0 \le \alpha < 1$

$$\left|\int_{-1}^{x} p\,(\alpha,t)\ \omega_n'(t)\ dt\right| \le C_{24}\ n^{1+2\alpha} .$$

The proof of lemma 3.6.3 can be given on the same lines as lemma 3.6.1.

<u>Lemma 3.6.4</u>. <u>We have for</u> $-1 \le x \le 1$, <u>and</u> $0 \le \alpha < 1$.

$$\left|\int_{-1}^{x} (3\alpha+t)\ p(\alpha,t)\ell_\nu'(t)\ dt\right| \le \frac{C_{25}\ n^{5/2+2\alpha}}{(1-x_\nu^2)^{(5+2\alpha)/4}[\omega_n'(x_\nu)]^2} .$$

The proof of this lemma too can be given on the same lines as lemma 3.6.2.

<u>Lemma 3.6.5</u>. <u>We have for</u> $-1+\varepsilon \le x \le 1-\varepsilon$ <u>and</u> $\alpha = 1/3$,

$$|r_0(x)| \le C_{26}\ n^{1/2} \tag{3.6.3}$$

<u>and</u>

$$|r_{n+1}(x)| \le C_{27}\ n^{1/2} . \tag{3.6.4}$$

<u>Proof</u>. Using lemmas 3.6.1, 3.6.3, 3.5.4 and (3.3.6) for $\alpha = 1/3$ we have from (3.2.11) and (3.2.12)

$$\begin{aligned} |r_0(x)| &\le C_{28}\ n^{-5/3} + C_{29}\ n^{-2/3} + C_{30}\ n^{1/2} \\ &\le C_{31}\ n^{1/2} \end{aligned}$$

from which (3.6.3) follows. (3.6.4) can be obtained similarly.

<u>Lemma 3.6.6</u>. <u>We have for</u> $-1+\varepsilon \le x \le 1-\varepsilon$ <u>and</u> $\alpha = 1/3$

$$\sum_{\nu=1}^{n} |r_\nu(x)| \le C_{32}\ n^{13/6} .$$

<u>Proof</u>. Since from (3.2.8) for $\alpha = 1/3$

$$|r_\nu(x)| \le \left| \frac{1-x^2}{1-x_\nu^2} \ell_\nu^2(x) \right| + \left| \frac{(1-x^2)\omega_n(x)\ell_\nu'(x)}{2(1-x_\nu^2)\omega_n'(x_\nu)} \right|$$

$$+ \frac{C_{33}\ \omega_n(x)}{(1-x_\nu^2)|\omega_n'(x_\nu)|} \left[\left| B_\nu \int_{-1}^{x} p(1/3,t)\omega_{n,1/3}(t)\ dt \right| \right.$$

$$\left. + \left| \int_{-1}^{x} (1+t)\ p(1/3,t)\ell'(t)\ dt \right| \right] + \left| C_\nu\ \rho_\nu(x) \right|$$

therefore

$$\sum_{\nu=1}^{n} |r_\nu(x)| \le \sum_{\nu=1}^{n} \frac{(1-x^2)}{(1-x_\nu^2)} [\ell_\nu(x)]^2 + \sum_{\nu=1}^{n} \frac{(1-x_\nu^2)|\omega_n(x)||\ell_\nu'(x)|}{2(1-x_\nu^2)|\omega_n'(x_\nu)|} \tag{3.6.5}$$

$$+ C_{33}\ |\omega_n(x)| \sum_{\nu=1}^{n} \frac{|B_\nu|}{(1-x_\nu^2)|\omega_n'(x_\nu)|}$$

$$\left| \int_{-1}^{x} p(1/3,t)\omega_{n,1/3}(t)\ dt \right|$$

$$+ C_{33}\ |\omega_n(x)| \sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)|\omega_n'(x_\nu)|} \left| \int_{-1}^{x} p(1/3,t)\ell_\nu'(t)\ dt \right|$$

$$+ \sum_{\nu=1}^{n} |C_\nu\ \rho_\nu(x)|$$

$$= k_1 + k_2 + k_3 + k_4 + k_5 \ .$$

Since

$$k_1 = \sum_{\nu=1}^{n} \frac{(1-x^2)\ell_\nu^2(x)}{(1-x_\nu^2)}$$

therefore using (3.3.1), (3.3.7), (3.3.8) and (3.3.10) we have

$$k_1 \le C_{34}\ n \ . \tag{3.6.6}$$

Since

$$k_2 = \sum_{\nu=1}^{n} \frac{(1-x_\nu^2)|\omega_n(x)||\ell_\nu'(x)|}{2(1-x_\nu^2)|\omega_n'(x_\nu)|}$$

therefore with the help of (3.3.11), (3.3.1), (3.3.6),(3.3.7) and (3.3.8) we have

(3.6.7) $$k_2 \leq C_{35}\, n$$

Again since

$$k_3 = C_{33}\, |\omega_n(x)| \sum_{\nu=1}^{n} \frac{|B_\nu|}{(1-x_\nu^2)|\omega_n'(x_\nu)|} \left|\int_{-1}^{1} p(1/3,t)\omega_{n,1/3}(t)\, dt\right|$$

therefore making use of (3.3.1), (3.3.6), (3.3.7), (3.3.8), lemma 3.5.1, lemma 3.5.4, lemma 3.6.2 and lemma 3.6.4, we have

(3.6.8) $$k_3 \leq C_{36}\, n^{13/6} .$$

Further since

$$k_4 = C_{33}\, |\omega_n(x)| \sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)|\omega_n'(x_\nu)|} \left|\int_{-1}^{1} p(1/3,t)\ell_\nu'(t)\, dt\right|$$

therefore using (3.3.1), (3.3.7), (3.3.6), (3.3.8) and lemma 3.6.4, we have

(3.6.9) $$k_4 \leq C_{37}\, n^{13/6} .$$

Finally since

$$k_5 = \sum_{\nu=1}^{n} |C_\nu\, \rho_\nu(x)$$

therefore using (3.3.1), (3.3.7), (3.3.8),(3.3.6), lemma 3.5.1, lemma 3.5.3, lemma 3.5.4 and lemma 3.5.5, we have

(3.6.10) $$k_5 \leq C_{38}\, n^{13/6} .$$

Hence from (3.6.5), (3.6.6), (3.6.7), (3.6.8), (3.6.9) and (3.6.10) the lemma follows.

7. Proof of the Theorem 3.2.3. Due to uniqueness theorem we have

$$\Phi_n(x) = \sum_{\nu=0}^{n+1} \Phi_n(x_\nu)\, r_\nu(x) + \sum_{\nu=1}^{n} \Phi_n''(x_\nu)\, \rho_\nu(x) \tag{3.7.1}$$

where $\Phi_n(x)$ is defined by the lemma 2.3.1 and by (3.2.3) we have

$$R_n(x;f) = \sum_{\nu=0}^{n+1} f(x_\nu)\, r_\nu(x) + \sum_{\nu=1}^{n} f''(x_\nu)\, \rho_\nu(x) \quad . \tag{3.7.2}$$

Since

$$|R_n(x;f) - f(x)| \le |R_n(x;f) - \Phi_n(x)| + |\Phi_n(x) - f(x)| \tag{3.7.3}$$

then using (3.7.1) and (3.7.2) we have

$$\begin{aligned} |R_n(x;f) - \phi_n(x)| &\le \sum_{\nu=0}^{n+1} f(x_\nu) - \Phi_n(x_\nu)|\ |r_\nu(x)| \\ &\quad + \sum_{\nu=1}^{n} f''(x_\nu) - \Phi_n''(x_\nu)|\ |\rho_\nu(x)| \\ &= |s_1| + |s_2| \quad . \end{aligned} \tag{3.7.4}$$

Now using lemma 2.3.1, lemma 3.6.5 and lemma 3.6.6, we have

$$\begin{aligned} |s_1| &\le C_{39} \frac{n^{13/6}}{n^{2+\mu}} \\ &= C_{39}\, n^{1/6-\mu} \quad . \end{aligned} \tag{3.7.5}$$

Further from lemma 3.5.6 and lemma 2.3.1, we have

$$\begin{aligned} |s_2| &\le C_{40} \frac{n^{1/6}}{n^{\mu}} \\ &= C_{40}\, n^{1/6-\mu} \quad . \end{aligned} \tag{3.7.6}$$

From (3.7.3)-(3.7.6) and lemma 2.3.1 we have if $\mu > 1/6$

$$|R_n(x;f) - f(x)| = o(1) \ .$$

This completes theproof of the Theorem.

CHAPTER IV

BALÁZS-TYPE INTERPOLATION ON JACOBI ABSCISSAS

1. In this chapter we consider the problems of existence, uniqueness and of explicit representation for polynomials $R_n(x)$ of degree $2n$ for $(0,2)$ interpolation of Balázs type when the nodes are the zeros of $\omega_n(x) \equiv P_n^{(\alpha,-\alpha)}(x)$, the n^{th} degree Jacobi polynomial. We also prove here a convergence theorem for $R_n(x)$ on the same lines as in [3].

2. Let

$$\rho(x) = (1-x)^{(1+\alpha)/2} \; (1+x)^{(1-\alpha)/2} \tag{4.2.1}$$

where $\alpha > -1$ and

$$-1 < x_n < x_{n-1} < \dots < x_1 < +1 \tag{4.2.2}$$

where x_ν $(\nu = 1, 2, \dots, n)$ are the zeros of $\omega_n(x) \equiv P_n^{(\alpha,-\alpha)}(x)$.

If now (x) of (4.2.1) is the weight function, (4.2.2) are the nodes and a_ν, b_ν are the arbitrary given numbers then we shall be looking for a polynomial $R_n(x)$ of degree $2n$ for which

$$R_n(x_\nu) = a_\nu \; ; \quad \{\rho(x)\, R_n(x)\}''_{x=x_\nu} = b_\nu \; , \quad (\nu = 1, 2, \dots, n) \tag{4.2.3}$$

and also

$$R_n(0) = \sum_{\nu=1}^{n} a_\nu \, \ell_\nu^2(0) \; , \tag{4.2.4}$$

where $\ell_\nu(x)$ is given by (3.2.5).

One can at once see from (3.2.5) that

$$\ell_\nu(x_j) = \begin{matrix} 0 & \text{if} & j \neq \nu \\ 1 & \text{if} & j = \nu \end{matrix} \quad . \tag{4.2.5}$$

Let $r_\nu(x)$ denote 2n-degree polynomial for which

$$r_\nu(x_j) = \begin{matrix} 0 & \text{if} & j \neq \nu \\ 1 & \text{if} & j = \nu \end{matrix} \quad ; \tag{4.2.6}$$

$$\{\rho(x)\, r_\nu(x)\}''_{x=x_j} = 0 \quad , \qquad (1 \leq \nu \leq n \,; 1 \leq j \leq n) \tag{4.2.7}$$

and let $\sigma_\nu(x)$ denote 2n-degree polynomial for which

$$\sigma_\nu(x_j) = 0 \;;\quad \{\rho(x)\, \sigma_\nu(x)\}''_{x=x_j} = \begin{matrix} 0 & \text{if} & j \neq \nu \\ 1 & \text{if} & j = \nu \end{matrix} \,, \qquad (1 \leq \nu \leq n \;;\; 1 \leq j \leq n). \tag{4.2.8}$$

Then (4.2.3) will be satisfied if we write

$$R_n(x) = \sum_{\nu=1}^{n} a_\nu\, r_\nu(x) + \sum_{\nu=1}^{n} b_\nu\, \sigma_\nu(x) \tag{4.2.9}$$

and if

$$\begin{aligned} R_n(0) &= \sum_{\nu=1}^{n} a_\nu\, r_\nu(0) + \sum_{\nu=1}^{n} b_\nu\, \sigma_\nu(0) \\ &= \sum_{\nu=1}^{n} a_\nu\, \ell_\nu^2(0) \;. \end{aligned} \tag{4.2.10}$$

We shall prove the following theorems:

<u>Theorem 4.2.1</u>. <u>If</u> $\rho(x)$ <u>and</u> $\{x_\nu\}_1^n$ <u>are given by</u> (4.2.1) <u>and</u> (4.2.2) <u>respectively and</u> a_ν, b_ν <u>are any arbitrary given values then there exists one and only one polynomial</u> $R_n(x)$ <u>of degree</u> 2n <u>which satisfies</u> (4.2.3) <u>and</u> (4.2.4). <u>The polynomial</u> $R_n(x)$ <u>of degree</u> 2n <u>can be written in the form</u> (4.2.9) <u>where the fundamental polynomials</u> $r_\nu(x)$ <u>and</u> $\sigma_\nu(x)$ <u>are given by</u>

$$r_\nu(x) = \ell_\nu^2(x) + \frac{\omega_n(x)}{\omega_n'(x_\nu)} \int_o^x \frac{\ell_\nu(t)(c_\nu t + d_\nu) - \ell_\nu'(t)}{t - x_\nu}\, dt \tag{4.2.11}$$

with

(4.2.12) $$c_\nu x_\nu + d_\nu = \ell'_\nu(x_\nu) ,$$

$$c_\nu = \frac{1-\alpha^2}{2(1-x_\nu^2)^2}$$

and

(4.2.13) $$\sigma_\nu(x) = \frac{\omega_n(x)}{2(1-x_\nu)^{(1+\alpha)/2}(1+x_\nu)^{(1-\alpha)/2}\omega'_n(x_\nu)} \int_o^x \ell_\nu(t)\, dt .$$

From (4.2.11) and (4.2.13) one can at once see that (4.2.4) holds.

Theorem 4.2.2. Let $f(x) \varepsilon C^1 [-1,1]$ and let $f'(x) \varepsilon \text{Lip}\, \mu$ $(0 < \mu \leq 1)$. Further let $0 < |\alpha| \leq 1/2$, $a_\nu = f(x_\nu)$ and $b_\nu = o(n^{3/4})(1-x_\nu)^{(\alpha-3)/2}(1+x_\nu)^{-(\alpha+3)/2}$, $(1 \leq \nu \leq n)$. Then the sequence of polynomials $R_n(x;f)$ given by (4.2.9) converges uniformly to $f(x)$ in $-1+\varepsilon \leq x \leq 1-\varepsilon$, $0 < \varepsilon < 1$ (ε being an arbitrary fixed positive number).

Remark. If $\alpha = 0$ and n is even the theorem is still ture.

3. In order to prove Theorem 4.2.1 and Theorem 4.2.2 we need the following lemmas.

Lemma 4.3.1. We have

(4.3.1) $$[\rho(x)\, \omega_n(x)]''_{x=x_j} = 0 , \qquad (1 \leq j \leq n) .$$

Proof. Since

$$[\rho(x)\, \omega_n(x)]''_{x_j} = [2\rho'(x_j)\, \omega'_n(x_j) + \rho(x_j)\, \omega''(x_j)] .$$

Therefore from (4.2.1) and (3.3.2), we have

$$[\rho(x)\ \omega_n(x)]''_{x_j} = \left[\frac{-2\rho\ (x_j)}{1-x_j^2}\ (\alpha+x_j)\ \omega'_n(x_j) + \rho(x_j)\ \omega''_n(x_j)\right]$$
$$= 0\ .$$

Lemma 4.3.2. *We have*

$$(\alpha+x)\ \ell_\nu(x) - (1-x^2)\ \ell'_\nu(x)$$
$$= (x-x_\nu)\ \Big[-(\alpha+x)\ \ell'_\nu(x) + \frac{1}{2}\ (1-x^2)\ \ell''_\nu(x)$$
$$+ \frac{n(n-1)}{2}\ \ell_\nu(x)\Big]\ .$$

Proof. From (3.2.5), we have

$$(4.3.2)\qquad (x-x_\nu)\ \ell_\nu(x) = \frac{\omega_n(x)}{\omega'_n(x_\nu)}\ , \qquad (1 \le \nu \le n)\ .$$

Differentiating both sides, using (3.3.2) and simplifying we have the lemma.

Lemma 4.3.3. *If* $\rho(x)$ *and* x_ν $(1 \le \nu \le n)$ *are given by* (4.2.1) *and* (4.2.2) *respectively then*

$$(4.3.3)\qquad \sigma_\nu(x) = \frac{\omega_n(x)\ (1+x_\nu)^{(\alpha-1)/2}}{2(1-x_\nu)^{(1+\alpha)/2}\omega'_n(x_\nu)}\ \int_o^x \ell_\nu(t)\ dt$$

is the polynomial of degree $2n$, *such that* (4.2.8) *is satisfied.*

Proof. With the help of (4.3.2) we can see that $\sigma_\nu(x)$ is a polynomial of degree $2n$. The first condition (4.2.8) is satisfied obviously since $\omega_n(x_j) = 0$ for all j , $1 \le j \le n$. From (4.3.3) we have

$$[\rho(x)\ \sigma_\nu(x)]''_{x_j} = A_{\nu,n}\ [\{\rho(x)\ \omega_n(x)\}''\ \int_o^x\ \ell_\nu(t)\ dt\ +$$

$$+ 2\{\rho'(x)\ \omega_n(x) + \rho(x)\ \omega_n'(x)\}\ \ell_\nu(x)$$

$$+ \omega_n(x)\ \rho(x)\ \ell_\nu'(x)]_{x_j} \quad ,$$

where

$$A_{\nu,n} = \frac{(1+x_\nu)^{(\alpha-1)/2}}{2(1-x_\nu)^{(1+\alpha)/2}\omega_n'(x_\nu)}$$

From (4.2.5) and lemma 4.3.1 , we have

$$[\rho(x)\ \sigma_\nu(x)]''_{x_j} = \begin{matrix} 0 & \text{if} & j \neq \nu \\ 1 & \text{if} & j = \nu \end{matrix} \quad .$$

This completes the proof of the lemma.

Lemma 4.3.4. *If* $\rho(x)$ *and* x_ν $(1 \leq \nu \leq n)$ *are given by* (4.2.1) *and* (4.2.2) *respectively then*

(4.3.4) $$r_\nu(x) = \ell_\nu^2(x) + \frac{\omega_n(x)}{\omega_n'(x_\nu)} \int_o^x \frac{\ell_\nu(t)(c_\nu t + d_\nu) - \ell_\nu'(t)}{t - x_\nu}\, dt$$

with

(4.3.5) $$c_\nu\ x_\nu + d_\nu = \ell_\nu'(x_\nu)$$

$$c_\nu = \frac{1 - \alpha^2}{2(1-x_\nu^2)^2} \quad ,$$

is a polynomial of degree $2n$, *such that* (4.2.6) *and* (4.2.7) *are satisfied.*

Proof. With the help of (4.3.2) and (3.3.2) one can see easily that $r_\nu(x)$ is a polynomial of degree $2n$. (4.2.6) follows from (4.3.2) and (4.2.2). To show that (4.3.4) satisfies condition (4.2.7) we proceed as follows:

$$(4.3.6)\qquad [\rho(x)r_\nu(x)]''_{x_j} = [\rho(x)\ell_\nu^2(x)]''_{x_j} + \left[\frac{\rho(x)\omega_n(x)}{\omega_n'(x_\nu)} \int_o^x \frac{\ell_\nu(t)(c_\nu t + d_\nu) - \ell_\nu'(t)}{t - x_\nu}\,dt\right]''_{x_j}$$

$$= [\,\rho''(x)\ell_\nu^2(x) + 4\rho'(x)\ell_\nu(x)\ell_\nu'(x) + 2\rho(x)\{\ell_\nu'^2(x) + \ell_\nu(x)\ell_\nu''(x)\}]_{x_j}$$

$$+ \frac{1}{\omega_n'(x_\nu)}\Bigg[\{\rho(x)\omega_n(x)\}'' \int_o^x \frac{\ell_\nu(t)(c_\nu t + d_\nu) - \ell_\nu'(t)}{t - x_\nu}\,dt$$

$$+ \rho(x)\omega_n(x)\left\{\frac{\ell_\nu'(x)(c_\nu x + d_\nu) + c_\nu\ell_\nu(x) - \ell_\nu''(x)}{x - x_\nu} - \frac{\ell_\nu(x)(c_\nu x + d_\nu) - \ell_\nu'(x)}{(x - x_\nu)^2}\right\}$$

$$+ 2\{\rho'(x)\omega_n(x) + \rho(x)\omega_n'(x)\}\frac{\ell_\nu(x)(c_\nu x + d_\nu) - \ell'(x)}{x - x_\nu}\Bigg]_{x_j} .$$

If $x_j \neq x_\nu$, then we have making use of lemma 4.3.1 and (4.2.1)

$$(4.3.7)\qquad [\rho(x)r_\nu(x)]''_{x_j} = 2\rho(x_j)\ell_\nu'^2(x_j) - \frac{2\rho(x_j)\omega_n'(x_j)\ell_\nu'(x_j)}{\omega_n'(x_\nu)(x_j - x_\nu)} .$$

From (4.3.7) and (4.3.2), we have

$$(4.3.8)\qquad [\rho(x)r_\nu(x)]''_{x_j} = 0 .$$

If $x_j = x_\nu$ we have from (4.3.6) making use of lemma 4.3.1

$$[\rho(x)r_\nu(x)]''_{x_\nu} = \rho''(x_\nu) + 4\rho'(x_\nu)\ell_\nu'(x_\nu) + 2\rho(x_\nu)\{\ell_\nu'^2(x_\nu) + \ell_\nu''(x_\nu)\}$$

$$+ \frac{1}{\omega_n'(x_\nu)}\left[2\rho(x_\nu)\omega_n'(x_\nu)\{\ell_\nu'(x_\nu)(c_\nu x_\nu + d_\nu) + c_\nu - \ell_\nu''(x_\nu)\}\right]$$

$$= \rho''(x_\nu) + 4\rho'(x_\nu)\ell_\nu'(x_\nu) + 4\rho(x_\nu)\,\ell_\nu'^2(x_\nu) + 2c_\nu\rho(x_\nu) .$$

From (4.2.1) and (4.3.2) we have

$$(4.3.10)\qquad [\rho(x)r_\nu(x)]''_{x_\nu} = \frac{(\alpha^2 - 1)\rho(x_\nu)}{(1 - x_\nu^2)^2} + \frac{(1 - \alpha^2)\rho(x_\nu)}{(1 - x_\nu^2)^2}$$

$$= 0 .$$

Hence from (4.3.8) and (4.3.10) we have (4.2.7). This completes the proof of the lemma.

Now we shall prove Theorem 4.2.1. According to lemma 4.3.3 and lemma 4.3.4, $R_n(x)$ defined by

(4.3.11) $$R_n(x) = \sum_{\nu=1}^{n} a_\nu r_\nu(x) + \sum_{\nu=1}^{n} b_\nu \sigma_\nu(x)$$

is a polynomial of degree $2n$ which satisifes (4.2.3) and (4.2.4) and which is identically equal to (4.2.9) with (4.2.11), (4.2.12) and (4.2.13). We have now to show that (4.3.11) is the only polynomial of degree $2n$ which satisfies Theorem 4.2.1.

Now suppose there exists another polynomial $S_n(x)$ which satisfies Theorem 4.2.1, i.e.

$$S_n(x_\nu) = a_\nu \; ; \; [\rho(x)S_n(x)]''_{x_\nu} = b_\nu \quad , \qquad (1 \leq \nu \leq n)$$

and

$$S_n(0) = \sum_{\nu=1}^{n} a_\nu \ell_\nu^2(0) \; .$$

Then

(4.3.12) $$R_n(x_\nu) - S_n(x_\nu) = 0 \; ; \; [\rho(x)\{R_n(x) - S_n(x)\}]''_{x_\nu} = 0 \; , \qquad (1 \leq \nu \leq n)$$

and

(4.3.13) $$R_n(0) - S_n(0) = 0 \; .$$

From (4.3.12) we have

$$R_n(x) - S_n(x) = \omega_n(x) \, g_n(x) \; ,$$

where $g_n(x)$ is any polynomial of degree n. Since $\omega_n(0) \neq 0$ therefore we have from (4.3.13),

(4.3.14) $$g_n(0) = 0 .$$

Now

$$\begin{aligned}[\rho(x)\{R_n(x)-S_n(x)\}]''_{x_\nu} &= [\rho(x)\omega_n(x)g_n(x)]''_{x_\nu} \\ &= [\rho(x)\omega_n(x)]''_{x_\nu} g_n(x_\nu) + \rho(x_\nu)\omega_n(x_\nu)g''_n(x_\nu) \\ &+ 2[\{\rho(x)\omega'_n(x) + \rho'(x)\omega_n(x)\}g'_n(x)]_{x_\nu} \\ &= 0 .\end{aligned}$$

From (4.2.2) and lemma 4.3.1, we have $g'_n(x_\nu) = 0$, $(1 \leq \nu \leq n)$. Therefore $g'_n(x) \equiv 0$ and we have $g_n(x) \equiv c$. But since $g_n(0) = 0$ by (4.3.14) hence we have $g_n(x) \equiv 0$ and $R_n(x) \equiv S_n(x)$, which completes the proof of Theorem 4.2.1.

4. In order to prove Theorem 4.2.2, we require the following lemmas.

Lemma 4.4.1 (J. Balázs). Let $f(x)$ be a differentiable function for $-1 \leq x \leq 1$ and $f'(x) \in \text{Lip}\,\mu$ $(0 < \mu \leq 1)$, then there exists a sequence of polynomials $\pi_{2n-2}(x)$ of degree $\leq 2n-2$, such that

(4.4.1) $$|\pi_{2n-2}(x) - f(x)| \leq \frac{k}{n^{1+\mu}}, \qquad (-1 \leq x \leq 1),$$

(4.4.2) $$|\pi'_{2n-2}(x)| \leq \frac{M_o}{(1-x^2)^{1/2}}, \qquad (-1 < x < 1)$$

and

$$|\pi''_{2n-2}(x)| \leq \frac{M_o}{(1-x^2)^{3/2}} + \frac{d_1 n^{1-\mu}}{1-x^2} , \qquad (-1 \leq x \leq 1) , \tag{4.4.3}$$

where

$$M_o = \max_{-1 \leq x \leq 1} |f'(x)| \quad \text{and} \quad d_1 \quad \text{is a constant.}$$

For the proof of the lemma we refer to Baláźs [3] pp. 330-335.

Lemma 4.4.2. If $q(x)$ is any polynomial of degree $\leq 2n$ then

$$q(x) \equiv \sum_{\nu=1}^{n} q(x_\nu) r_\nu(x) + \sum_{\nu=1}^{n} \{\rho(x) q(x)\}''_{x_\nu} \sigma_\nu(x) + c_n \omega_n(x) , \tag{4.4.4}$$

where

$$c_n = \frac{1}{\omega_n(0)} \left[q(0) - \sum_{\nu=1}^{n} q(x_\nu) \ell^2(0) \right] \tag{4.4.5}$$

Proof. Let

$$P(x) \equiv q(x) - \sum_{\nu=1}^{n} q(x_\nu) r_\nu(x) - \sum_{\nu=1}^{n} \{\rho(x) q(x)\}''_{x_\nu} \sigma_\nu(x) . \tag{4.4.6}$$

Since from (4.2.6) and (4.2.8) we have $P(x_j) = 0$, $(1 \leq j \leq n)$ therefore

$$P(x) = \omega_n(x)\, g_n(x) , \tag{4.4.7}$$

where $g_n(x)$ is any polynomial of degree n . From (4.4.6), (4.2.7), (4.2.3) and (4.3.2), we have

$$[\rho(x)\, P(x)]''_{x_j} = 0 = 2\rho(x_j)\omega'_n(x_j) g'_n(x_j) , \qquad j = 1,2,\dots n ,$$

which implies that $g'_n(x) \equiv 0$, therefore $g_n(x) \equiv c$, and hence from (4.4.6) and (4.4.7), we have

$$c_n\, \omega_n(x) = q(x) - \sum_{\nu=1}^{n} q(x_\nu) r_\nu(x) - \sum_{\nu=1}^{n} \{\rho(x) q(x)\}''_{x_\nu} \sigma_\nu(x) \tag{4.4.8}$$

from which (4.4.4) follows and (4.4.5) follows from (4.4.8) using (4.3.3) and (4.3.4).

Lemma 4.4.3. For $\alpha > -1$ the following estimates are valid

$$(4.4.9) \qquad \sum_{\nu=1}^{n} (1-x_\nu^2)^{-p} [\omega_n'(x_\nu)]^{-2} \leq K_1, \qquad p = 0, 1 ,$$

$$\leq K_2 \log n , p = 2,$$

$$(4.4.10) \qquad \sum_{\nu=1}^{n} (1-x_\nu^2)^{-(2\alpha+1)/4 -p} |\omega_n'(x_\nu)|^{-3} \leq K_3 n^{-1/2} , p = 0,2$$

and

$$(4.4.11) \qquad \sum_{\nu=1}^{n} (1-x_\nu^2)^{-(2\alpha+1)/2 -2} |\omega_n'(x_\nu)|^{-4} \leq K_4 n^{-1} ,$$

where K_i's , $1 \leq i \leq 4$ are constants.

Proof. (4.4.9)-(4.4.11) follow from (3.3.1), (3.3.7) and (3.3.8).

Lemma 4.4.4. For $0 \leq |\alpha| \leq 1/2$ we have

$$(4.4.12) \qquad |1 - \sum_{\nu=1}^{n} \ell_\nu^2(0)| = 0(n^{-1/2}) .$$

Proof. To prove (4.4.12) we first need to prove

$$(4.4.13) \qquad \sum_{\nu=1}^{n} \ell_\nu^2(0) = 0(1) .$$

Let δ be a fixed positive number $\delta < 1$. Then using (3.3.6) we have

$$\sum_{|x_\nu| > \delta}^{n} \ell_\nu^2(0) = 0(n^{-1}) \sum_{\nu=1}^{n} [\omega_n'(x_\nu)]^{-2} .$$

Consequently from lemma 4.4.3 we have

$$(4.4.14) \qquad \sum_{|x_\nu| > \delta}^{n} \ell_\nu^2(0) = 0(n^{-1}) .$$

Now assume $|x_\nu| \leq \delta$. For fixed ν we have from Szegö [31(1)] formula (14.4.5) pp. 334 ,

$$(4.4.15) \qquad \ell_\nu(0) = 0(n^{-1/2}) \frac{\omega_n(x_\nu) - \omega_n(0)}{x_\nu} = 0(1) .$$

Denoting $\theta_\nu = \text{arc cos } x_\nu$, $0 < \theta_\nu < \pi$, we have from Szegö [32(1)] pp 121 formula (6.21.5) ,

$$(4.4.16) \qquad \theta_{\nu+1} - \theta_\nu \geq \frac{2\nu+1}{2n+1}\pi - \frac{2\nu}{2n+1}\pi = \frac{\pi}{2n+1} .$$

Therefore from (4.4.16) and (4.4.15), we have

$$\sum_{|x_\nu|\leq\delta} [\ell_\nu^2(0)]^2 = \sum_{n^{-1}<|\theta_\nu-\pi/2|\leq\delta'} \ell_\nu^2(0) + 0(1) ,$$

where `δ' is a fixed positive number. Using (4.4.15),(3.3.6) and simplifying, we have

$$\sum_{|x_\nu|\leq\delta} \ell_\nu^2(0) = 0(n^{-1})\, 0(n^{-1}) \sum_{n^{-1}<|\theta_\nu-\pi/2|\leq\delta'} (\theta_\nu-\pi/2)^{-2} + 0(1) .$$

Finally, making use of the formula (8.9.1) Szegö [32(1)] pp. 236, we have

$$(4.4.17) \qquad \sum_{|x_\nu|\leq\delta} \ell_\nu^2(0) = 0(1) .$$

Hence from (4.4.14) and (4.4.17) we have (4.4.13).

Now we shall establish (4.4.12). Since from Szegö [32(1)] pp. 329, we have

$$\sum_{\nu=1}^{n} \left[1 - \frac{\omega_n''(x_\nu)}{\omega_n'(x_\nu)}(x-x_\nu)\right] \ell_\nu^2(x) = 1$$

therefore

$$|1 - \sum_{\nu=1}^{n} \ell_\nu^2(0)| \leq 2|\alpha| \sum_{\nu=1}^{n} \frac{|x_\nu|}{1-x_\nu^2} \ell_\nu^2(0) + 2 \sum_{\nu=1}^{n} \frac{x_\nu^2 \, \ell_\nu^2(0)}{1-x_\nu^2}$$

$$(4.4.18) \qquad = S_1^* + S_2^* .$$

With the help of (3.3.6) we have

$$S_1^* = 2|\alpha| \sum_{=1}^{n} \frac{|x_\nu|}{1-x_\nu^2} \ell_\nu^2(0)$$

(4.4.19) $$= 0(n^{-1/2}) \sum_{\nu-1}^{n} \frac{|\ell_\nu(0)|}{(1-x_\nu^2)\ \omega_n'(x_\nu)}$$

Making use of the Schwarz inequality in (4.4.19), we have

$$S_1^* = 0(n^{-1/2}) \left[\sum_{\nu=1}^{n} \ell_\nu^2(0)\right]^{1/2} \left[\sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)[\omega_n'(x_\nu)]^2}\right]^{1/2} .$$

Hence from (4.4.13) and lemma 4.4.3, we have

(4.4.20) $$S_1^* = 0(n^{-1/2}) .$$

Further using (3.3.6) and lemma 4.4.3, we have

(4.4.21) $$S_2^* = 0(n^{-1}) \sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)[\omega_n'(x_\nu)]^2}$$

$$= 0(n^{-1}) .$$

From (4.4.18), (4.4.20) and (4.4.21) we have the required result.

5. Estimation of the fundamental polynomials of second kind.

Lemma 4.5.1. We have for $-1 + \varepsilon \leq x \leq 1-\varepsilon$, $(0 < \varepsilon < 1)$

(4.5.1) $$\sum_{|x_\nu| \geq 1-\varepsilon/2} (1-x_\nu^2)^{-2}[|\omega_n'(x_\nu)|]^{-1}\left|\int_0^x \ell_\nu(t)\ dt\right| \leq c_1\ n^{-1/2} \log n$$

and

(4.5.2) $$\sum_{|x_\nu| < 1-\varepsilon/2} (1-x_\nu^2)^{-2} [|\omega_n'(x_\nu)|]^{-1}\left|\int_0^x \ell_\nu(t)\ dt\right| \leq c_2\ n^{-1/2} \log n ,$$

where c_j , $j = 1,2, \ldots$ are constants depending on α and ε .

Proof. Let

$$I_1 = \sum_{|x_\nu| \geq 1-\varepsilon/2} (1-x_\nu^2)^{-2} [|\omega_n'(x_\nu)|]^{-1} \left| \int_o^x \ell_\nu(t)\, dt \right|$$

and

$$I_2 = \sum_{|x_\nu| < 1-\varepsilon/2} (1-x_\nu^2)^{-2} [|\omega_n'(x_\nu)|]^{-1} \left| \int_o^x \ell_\nu(t)\, dt \right| .$$

Since $-1 + \varepsilon \leq x \leq 1-\varepsilon$, therefore $0 \leq t \leq x$ or $x \leq t \leq 0$ and if $|x_\nu| \geq 1-\varepsilon/2$, then $|t-x_\nu| > \varepsilon/2$ and $1 \leq \dfrac{2(1-t)^{\alpha/2} (1+t)^{-\alpha/2}}{\varepsilon^{\alpha/2}}$. Hence we have

$$I_1 \leq \sum_{|x_\nu| \geq 1-\varepsilon/2} \frac{4}{(1-x_\nu^2)^2 [\omega_n'(x_\nu)]^2} \left| \int_o^x \frac{\omega_n(t)(1-t)^{\alpha/2}}{\varepsilon^{(2+\alpha)/2} (1+t)^{\alpha/2}}\, dt \right|$$

$$\leq c_3 \sum_{|x_\nu| \geq 1-\varepsilon/2} \frac{1}{(1-x_\nu^2)^2 [\omega_n'(x_\nu)]^2} \int_o^x \frac{|\omega_n(t)| (1-t)^{\alpha/2}}{(1+t)^{\alpha/2}}\, dt .$$

Now applying the Schwarz's inequality, we have

$$I_1 \leq c_4 \left[\int_{-1}^{1} \{\omega_n(t)\}^2 (1-t)^\alpha (1+t)^{-\alpha}\, dt \right]^{1/2} \sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)^2 [\omega_n'(x_\nu)]^2} .$$

Consequently, using (3.3.4), (3.3.5) and lemma 4.4.3, we have

$$I_1 \leq c_4\, n^{-1/2} \log n ,$$

which completes the proof of (4.5.1) .

If we take $|x_\nu| < 1-\varepsilon/2$ then $(1-x_\nu^2) > \varepsilon(1-\varepsilon/4)$ and in $-1 + \varepsilon \leq x \leq 1-\varepsilon$, we have

$$I_2 = \sum_{|x_\nu|<1-\varepsilon/2} \frac{1}{(1-x_\nu^2)^2|\omega_n'(x_\nu)|} \left|\int_0^x \ell_\nu(t)\,dt\right|$$

$$\leq c_5 \sum_{|x_\nu|<1-\varepsilon/2} \frac{1}{|\omega_n'(x_\nu)|} \left|\int_0^x \ell_\nu(t)\,dt\right| .$$

Hence using (3.3.12) we have

(4.5.3) $$I_2 \leq c_5 \sum_{|x_\nu|<1-\varepsilon/2} \frac{\Gamma(n+\alpha+1)\,\Gamma(n-\alpha+1)}{(1-x_\nu^2)\{\Gamma(n+1)\}^2|\omega_n'(x_\nu)|^3}$$

$$\times \left[\sum_{j=0}^{n-1} \frac{(2j+1)\{\Gamma(j+1)\}^2|\omega_j(x_\nu)|}{\Gamma(j+\alpha+1)\,\Gamma(j+1-\alpha)} \left|\int_0^x \omega_j(t)\,dt\right|\right]$$

$$\leq c_5 \sum_{|x_\nu|<1-\varepsilon/2} \frac{\Gamma(n+\alpha+1)\,\Gamma(n-\alpha+1)}{(1-x_\nu^2)\{\Gamma(n+1)\}^2|\omega_n'(x_\nu)|^3}$$

$$\times \left[\sum_{j=0}^{n-1} \frac{2j\{\Gamma(j+1)\}^2|\omega_j(x_\nu)|}{\Gamma(j+\alpha+1)\,\Gamma(j-\alpha+1)} \left|\int_0^x \omega_j(t)\,dt\right|\right.$$

$$\left. + \sum_{j=0}^{n-1} \frac{\{\Gamma(j+1)\}^2|\omega_j(x_\nu)|}{\Gamma(j+\alpha+1)\,\Gamma(j-\alpha+1)} \left|\int_0^x \omega_j(t)\,dt\right|\right]$$

$$= r_1^* + r_2^* .$$

First we shall find the estimate for r_2^* .

Since

$$r_2^* = c_5 \sum_{|x_\nu|<1-\varepsilon/2} \frac{\Gamma(n+\alpha+1)\ \Gamma(n-\alpha+1)}{(1-x_\nu^2)\{\Gamma(n+1)\}^2|\omega_n'(x_\nu)|^3}$$

$$\times \left[\sum_{j=0}^{n-1} \frac{\{\Gamma(j+1)\}^2|\omega_j(x_\nu)|}{\Gamma(j+\alpha+1)\ \Gamma(j-\alpha+1)} \left| \int_o^x \omega_j(t)\ dt \right| \right]$$

therefore with the help of (3.3.5), (3.3.6), (3.3.9) and lemma 4.4.3, we have

(4.5.4) $$r_2^* \leq c_6\ n^{-1} \log n\ .$$

Further since

$$r_1^* = c_5 \sum_{|x_\nu|<1-\varepsilon/2} \frac{2\ \Gamma(n+\alpha+1)\ \Gamma(n-\alpha+1)}{(1-x_\nu^2)\{\Gamma(n+1)\}^2|\omega_n'(x_\nu)|^3}$$

$$\times \left[\sum_{j=0}^{n-1} \frac{\{\Gamma(j+1)\}^2|\omega_j(x_\nu)|}{\Gamma(j+\alpha+1)\ \Gamma(j-\alpha+1)} \left| \int_o^x j\omega_j(t)\ dt \right| \right]$$

therefore using (3.3.2) we have

$$r_1^* = c_5 \sum_{|x_\nu|<1-\varepsilon/2} \frac{2\ \Gamma(n+\alpha+1)\ \Gamma(n-\alpha+1)}{(1-x_\nu^2)\{\Gamma(n+1)\}^2|\omega_n'(x_\nu)|^3}$$

$$\times \left[\sum_{j=0}^{n-1} \frac{\{\Gamma(j+1)\}^2|\omega_j(x_\nu)|}{\Gamma(j+\alpha+1)\ \Gamma(j-\alpha+1)} \left| \int_o^x [2\alpha\omega_j(t)-(1-t^2)\omega_j'(t)]'\,dt \right| \right]$$

$$\leq c_7 \sum_{|x_\nu|<1-\varepsilon/2} \frac{\Gamma(n+\alpha+1)\ \Gamma(n-\alpha+1)}{(1-x_\nu^2)\{\Gamma(n+1)\}^2|\omega_n'(x_\nu)|^3}$$

$$\times \left[\sum_{j=0}^{n-1} \frac{2\{\Gamma(j+1)\}^2 |\omega_j(x_\nu)|}{\Gamma(j+\alpha+1)\Gamma(j-\alpha+1)(j+1)} \left\{ 2|\alpha|\omega_j(x)| + 2\alpha|\omega_j(0)| + |(1-x^2)\omega_j'(x)| + |\omega_j'(0)| \right\} \right]$$

Making use of (3.3.3) we have

$$r_1^* \le c_8 \sum_{|x_\nu|<1-\varepsilon/2} \frac{2\Gamma(n+\alpha+1)\ \Gamma(n-\alpha+1)}{\{\Gamma(n+1)\}^2 |\omega_n'(x_\nu)|^3} \sum_{j=0}^{n-1} \frac{\{\Gamma(j+1)\}^2 |\omega_j(x_\nu)|}{\Gamma(j+\alpha+1)\Gamma(j-\alpha+1)(j+1)}$$

$$\times \left| 3|\alpha|\omega_j(x)| + (j+1)|\omega_j(x)| + (j+1)|\omega_{j+1}(0)| + 3|\alpha|\omega_j(0)| \right]$$

Hence with the help of (3.3.5), (3.3.6), (3.3.9) and lemma 4.4.3, we have

$$r_1^* \le c_9\, n^{-1/2} \log n\ . \tag{4.5.5.}$$

So from (4.5.4), (4.5.3) and (4.4.5), we have

$$I_2 \le c_{10}\, n^{-1/2} \log n\ ,$$

which completes the proof of (4.5.2) .

Lemma 4.5.2. If $-1 + \varepsilon \le x \le 1-\varepsilon$, $(0 < \varepsilon < 1)$ then we have

$$\sum_{\nu=1}^{n} (1-x_\nu)^{(\alpha-3)/2} (1+x_\nu)^{-(\alpha-3)/2} |\sigma_\nu(x)| \le c_{11}\, n^{-1} \log n\ , \tag{4.5.6}$$

where $\sigma_\nu(x)$ is given by (4.3.3) and c_{11} is a constant depending on α and ε .

Proof. From (4.3.3) we have

$$\sum_{\nu=1}^{n} \frac{(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}} |\sigma_\nu(x)| = \frac{|\omega_n(x)|}{2} \sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)^2 |\omega_n'(x_\nu)|} \left| \int_0^x \ell_\nu(t)\, dt \right| \tag{4.5.7}$$

$$= \frac{|\omega_n(x)|}{2} \left\{ \sum_{|x_\nu|\ge 1-\varepsilon/2} + \sum_{|x_\nu|<1-\varepsilon/2} \right\} .$$

From (4.5.7), (3.3.6) and lemma 4.5.1 the lemma follows.

Estimation of the fundamental polynomials of the first kind.

Consider

$$(4.5.8)\qquad \frac{\omega_n(x)}{\omega_n'(x_\nu)}\int_o^x \frac{\ell_\nu(t)\ (c_\nu t + d_\nu)-\ell_\nu'(t)}{t - x_\nu}\,dt$$

$$= \frac{c_\nu\omega_n(x)}{\omega_n'(x_\nu)}\int_o^x \ell_\nu(t)\ dt + \frac{\omega_n(x)}{\omega_n'(x_\nu)}\int_o^x \frac{(c_\nu x_\nu+d_\nu)\ell_\nu(t)-\ell_\nu'(t)}{t - x_\nu}\,dt\ .$$

From (4.5.8), (4.3.2) and lemma 4.3.2, we have after long computations

$$(4.5.9)\qquad \frac{\omega_n(x)}{\omega_n'(x_\nu)}\int_o^x \frac{(c_\nu x_\nu+ d_\nu)\ell_\nu(t)-\ell_\nu'(t)}{t - x_\nu}\,dt$$

$$= \frac{(\alpha+ x_\nu)\ell_\nu(0)\omega_n(x)}{(1-x_\nu^2)\omega_n'(x_\nu)} - \frac{(\alpha+x+x_\nu)\ell_\nu(x)\omega_n(x)}{(1-x_\nu^2)\omega_n'(x_\nu)}$$

$$+ \frac{(1-x^2)\ell_\nu'(x)\omega_n(x)}{2(1-x_\nu^2)\omega_n'(x_\nu)} - \frac{\ell_\nu'(0)\omega_n(x)}{2(1-x_\nu^2)\,\omega_n'(x_\nu)}$$

$$+ \frac{n(n+1)\omega_n(x)}{2(1-x_\nu^2)\omega_n'(x_\nu)}\int_o^x \ell_\nu(t)\ dt\ .$$

From (4.5.8) and (4.2.11) we have

$$r_\nu(x) = \ell_\nu^2(x) + \frac{(\alpha+x_\nu)\ell_\nu(0)\omega_n(x)}{(1-x_\nu^2)\omega_n'(x_\nu)}$$

$$(4.5.10)\qquad - \frac{(\alpha+x_\nu+x)\ell_\nu(x)\omega_n(x)}{(1-x_\nu^2)\omega_n'(x_\nu)} + \frac{(1-x^2)\ell_\nu'(x)\omega_n(x)}{2(1-x_\nu^2)\omega_n'(x_\nu)}$$

$$- \frac{\ell_\nu'(0)\omega_n(x)}{2(1-x_\nu^2)\omega_n'(x_\nu)} + \frac{[n(n+1)(1-x_\nu^2)+(1-\alpha_\nu^2)]\omega_n(x)}{2(1-x_\nu^2)^2\omega_n'(x_\nu)}\int_o^x \ell_\nu(t)\ dt\ .$$

<u>Lemma 4.5.3</u>. For $-1+\varepsilon \leq x \leq 1-\varepsilon$, $(0 < \varepsilon < 1)$ we have

$$\sum_{\nu=1}^{n} |r_\nu(x)| \leq c_{12}\, n \log n \,. \tag{4.5.11}$$

<u>Proof</u>. From (4.5.10) we have

$$|r_\nu(x)| \leq \ell_\nu^2(x) + \frac{|(\alpha+x_\nu)\ell_\nu(0)\omega_n(0)|}{(1-x_\nu^2)|\omega_n'(x_\nu)|}$$

$$+ \frac{|(\alpha+x+x_\nu)\ell_\nu(x)\omega_n(x)|}{(1-x_\nu^2)|\omega_n'(x_\nu)|} + \frac{(1-x_\nu^2)|\ell_\nu'(x)\omega_n(x)|}{2(1-x_\nu^2)|\omega_n'(x_\nu)|} \tag{4.5.12}$$

$$+ \frac{|\ell_\nu(0)\omega_n(x)|}{2(1-x_\nu^2)|\omega_n'(x_\nu)|} + \left| \frac{[n(n+1)(1-x_\nu^2)+(1-\alpha^2)]\omega_n(x_\nu)}{2(1-x_\nu^2)^2\omega_n'(x_\nu)} \int_o^x \ell_\nu(t)\,dt \right| .$$

Summing both sides of (4.5.12), we have

$$\sum_{\nu=1}^{n} |r_\nu(x)| \leq \sum_{\nu=1}^{n} \ell_\nu^2(x) + \sum_{\nu=1}^{n} \frac{|\ell_\nu(0)|\ |(\alpha+x_\nu)|\ |\omega_n(x)|}{(1-x_\nu^2)|\omega_n'(x_\nu)|}$$

$$+ \sum_{\nu=1}^{n} \frac{|\alpha+x+x_\nu|\ |\ell_\nu(x)|\ |\omega_n(x)|}{(1-x_\nu^2)|\omega_n'(x_\nu)|} \tag{4.5.13}$$

$$+ \sum_{\nu=1}^{n} \frac{(1-x^2)|\ell_\nu'(x)|\ |\omega_n(x)|}{2(1-x_\nu^2)|\omega_n'(x_\nu)|}$$

$$+ \sum_{\nu=1}^{n} \frac{|\ell_\nu'(0)|\ |\omega_n(x)|}{2(1-x_\nu^2)|\omega_n'(x_\nu)|}$$

$$+ \sum_{\nu=1}^{n} \frac{|[n(n+1)(1-x_\nu^2)+(1-\alpha^2)]\omega_n(x)|}{2(1-x_\nu^2)^2|\omega_n'(x_\nu)|} \left|\int_o^x \ell_\nu(t)\,dt\right|$$

$$= p_1 + p_2 + p_3 + p_4 + p_5 + p_6 \,.$$

Since

$$p_1 = \sum_{=1}^{n} \ell_\nu^2(x)$$

therefore using (3.3.10) we have

$$p_1 = 0(n^2) \sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)^{(5+2\alpha)/2} [\omega_n'(x_\nu)]^4} \ .$$

With the help of lemma 4.4.3, we have

$$(4.5.14) \qquad p_1 = 0(n) \ .$$

Since

$$p_2 = \sum_{\nu=1}^{n} \frac{|\alpha+x_\nu)| \ |\ell_\nu(0)| \ |\omega_n(x)|}{(1-x_\nu^2)|\omega_n'(x_\nu)|}$$

therefore using (3.3.6), (3.3.10) and lemma 4.4.3, we have

$$(4.5.15) \qquad p_2 = 0(1) \ .$$

Since

$$p_3 = \sum_{\nu=1}^{n} \frac{|(\alpha+x+x_\nu)| \ |\ell_\nu(x)| \ |\omega_n(x)|}{(1-x_\nu^2)|\omega_n'(x_\nu)|}$$

therefore making use of (3.3.6), (3.3.10) and lemma 4.4.3, we have

$$(4.5.16) \qquad p_3 = 0(1) \ .$$

Further since

$$p_4 = \sum_{\nu=1}^{n} \frac{(1-x^2)|\ell_\nu'(x)|\ |\omega_n(x)|}{2(1-x_\nu^2)|\omega_n'(x_\nu)|}$$

therefore, with the help of (3.3.6), (3.3.11) and lemma 4.4.3 one obtains

(4.5.17) $$p_4 = 0(n)\ .$$

Similarly

(4.5.18) $$p_5 = 0(n)\ .$$

Now since

(4.5.19) $$p_6 = \sum_{\nu=1}^{n} \frac{|[n(n+1)(1-x_\nu^2)+(1-\alpha^2)]\omega_n(x)|}{2(1-x_\nu^2)^2|\omega_n'(x_\nu)|} \left|\int_0^x \ell_\nu(t)\ dt\right|$$

$$= 0(n^2) \sum_{\nu=1}^{n} \frac{|\omega_n(x)|}{2(1-x_\nu^2)^2|\omega_n'(x_\nu)|} \left|\int_0^x \ell_\nu(t)\ dt\right|$$

$$= 0(n^2)\ \frac{|\omega_n(x)|}{2} \left\{ \sum_{|x_\nu|\geq 1-\varepsilon/2} + \sum_{|x_\nu|<1-\varepsilon/2} \right\}$$

From (4.5.19), (3.3.6) and lemma 4.5.1, we have

(4.5.20) $$p_6 = 0(n \log n)\ .$$

Now from (4.5.14), (4.5.15), (4.5.16), (4.5.17), (4.5.18), (4.5.20) and (4.5.13) the lemma follows.

6. <u>Proof of Theorem 4.2.2</u>.

From lemma 4.4.2, we have

$$\Pi_{2n-2}(x) \equiv \sum_{\nu=1}^{n} \pi_{2n-2}(x_\nu) r_\nu(x) + \sum_{\nu=1}^{n} \left[\sigma(x)\ \pi_{2n-2}(x)\right]''_{x_\nu} \sigma_\nu(x)$$

$$+ \ d_n\ \omega_n(x)\ . \tag{4.6.1}$$

where d_n is a constant and is given by

$$d_n = \frac{1}{\omega_n(0)} \left[\pi_{2n-2}(0) - \sum_{\nu=1}^{n} \pi_{2n-2}(x_\nu) \ell^2(0)\right] \tag{4.6.2}$$

From (4.6.1) and (4.2.9) we have if $a_\nu = f(x_\nu)$ and $b_\nu = \dfrac{o(n^{3/4})(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}}$,

$(1 \leq \nu \leq n)$,

$$|f(x) - R_n(x; f)| \leq |f(x) - \pi_{2n-2}(x)| + \sum_{\nu=1}^{n} |\pi_{2n-2}(x_\nu) - f(x_\nu)|\,|r_\nu(x)|$$

$$+ \sum_{\nu=1}^{n} \left| [\rho(x)\ \pi_{2n-2}(x)]''_{x_\nu} - o(n^{3/4}) \frac{(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}} \right| |\sigma_\nu(x)|$$

$$+ \ |\ d_n\ \omega_n(x)|\ .$$

From lemma 4.4.1 and lemma 4.5.3, we have

$$|f(x) - R_n(x;f)| \leq \frac{k}{n^{1+\mu}} + \frac{k}{n^{1+\mu}} [c_{12}\ n \log n]$$

$$+ \sum_{\nu=1}^{n} \left| [\rho(x)\ \pi_{2n-2}(x)]''_{x_\nu} - o(n^{3/4}) \frac{(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}} \right| |\sigma_\nu(x)| \tag{4.6.3}$$

$$+ \ |\ d_n\ \omega_n(x)|\ .$$

Now

$$\sum_{\nu=1}^{n} \left| [\rho(x)\ \pi_{2n-2}(x)]''_{x_\nu} - o(n^{3/4}) \frac{(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}} \right| |\sigma_\nu(x)|$$

$$\leq \sum_{\nu=1}^{n} |\rho''(x_\nu)\ \pi_{2n-2}(x_\nu)|\ |\sigma_\nu(x)| + 2\sum_{\nu=1}^{n} |\rho'(x_\nu)\pi'_{2n-2}(x_\nu)|\,|\sigma_\nu(x)|$$

$$+ \sum_{\nu=1}^{n} |\rho(x_\nu)\pi''_{2n-2}(x_\nu)|\,|\sigma_\nu(x)| + o(n^{3/4}) \sum_{\nu=1}^{n} \frac{(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}} |\sigma_\nu(x)|$$

$$= v_1 + v_2 + v_3 + v_4 \quad \text{(say)} .$$

Since from (4.2.1), $\rho''(x_\nu) = \dfrac{(\alpha^2 - 1)}{(1-x_\nu^2)^2}\ \rho(x_\nu)$ and from lemma 4.4.1 ,

$|\pi_{2n-2}(x)| \leq k_1$, where k_1 is a constant therefore

$$v_1 \leq c_{13} \sum_{\nu=1}^{n} \frac{(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}} |\sigma_\nu(x)| .$$

Making use of lemma 4.5.2 , we have

$$v_1 \leq c_{14}\ n^{-1} \log n . \tag{4.6.4}$$

From (4.2.1) and (4.4.2) we have using lemma 4.5.2 ,

$$v_2 \leq c_{15} \sum_{\nu=1}^{n} \frac{(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}} |\sigma_\nu(x)| \leq c_{16}\ n^{-1} \log n . \tag{4.6.5}$$

From (4.2.1), (4.4.3) and lemma 4.5.2 we have, since $1-x_\nu^2 < 1$, for $-1 + \varepsilon \leq x \leq 1-\varepsilon$,

(4.6.6) $$v_3 \le \sum_{\nu=1}^{n} |\rho(x_\nu)| \left[\frac{M_o}{(1-x_\nu^2)^{3/2}} + \frac{d_1 n^{1-\mu}}{(1-x_\nu^2)} \right] |\sigma_\nu(x)|$$

$$\le c_{17} \sum_{\nu=1}^{n} \frac{(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}} |\sigma_\nu(x)|$$

$$+ c_{18}\, n^{1-\mu} \sum_{\nu=1}^{n} \frac{(1-x_\nu)^{(\alpha-3)/2}}{(1+x_\nu)^{(\alpha+3)/2}} |\sigma_\nu(x)|$$

$$\le c_{19}\, n^{-\mu} \log n .$$

Again using lemma 4.5.2, we have for $-1 + \varepsilon \le x \le 1-\varepsilon$,

(4.6.7) $$v_4 \le c_{12}\, n^{-1/4} \log n .$$

Hence from (4.6.4), (4.6.5), (4.6.6), (4.6.3) and (4.6.7) we have for $-1 + \varepsilon \le x \le 1-\varepsilon$ and $0 < \mu \le 1$,

(4.6.8) $$|f(x) - R_n(x;f)| \le c_{21} \log n\, [n^{-\mu} + n^{-1/4}] + |d_n\, \omega_n(x)| .$$

Since from (3.3.14)

$$\omega_n(0) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n,\ 1+n; \\ 1+\alpha; \end{matrix}\ 1/2\right]$$

therefore using (3.3.5) and problem 3, Rainville [23] pp. 69 we have

(4.6.9) $$|\omega_n(0)| > c\, n^{-1/2} ,$$

where c is a constant.

From (4.6.2) and (4.6.9) we have

$$(4.6.10) \qquad |d_n\, \omega_n(x)| = \left|\frac{\omega_n(x)}{\omega_n(0)}\right| \left|\pi_{2n-2}(0) - \sum_{\nu=1}^{n} \pi_{2n-2}(x_\nu)\ell_\nu^2(0)\right|$$

$$\leq c_{22} \left|\pi_{2n-2}(0) - \sum_{\nu=1}^{n} \pi_{2n-2}(x_\nu)\ell_\nu^2(0)\right|$$

$$= c_{22}\, u_1 \quad \text{(say)},$$

where

$$u_1 = \left|\pi_{2n-2}(0) - \sum_{\nu=1}^{n} \pi_{2n-2}(x_\nu)\ell_\nu^2(0)\right|$$

$$= \Big| \sum_{\nu=1}^{n} [\pi_{2n-2}(0) - \pi_{2n-2}(x_\nu)]\ell^2(0)$$

$$+ \pi_{2n-2}(0)\, [1 - \sum_{\nu=1}^{n} \ell_\nu^2(0)]\Big| \; .$$

Using lemmas 4.4.1 and 4.4.4, we have

$$u_1 \leq \sum_{\nu=1}^{n} |\pi_{2n-2}(0) - \pi_{2n-2}(x_\nu)|\ell^2(0) + c_{23}\, n^{-1/2}$$

$$\leq \sum_{\nu=1}^{n} |x_\nu|\, |\pi'_{2n-2}(\eta_\nu)|\, \ell_\nu^2(0) + c_{23}\, n^{-1/2} ,$$

where $0 \leq |\eta_\nu| \leq x_\nu|$.

Since $1-x_\nu^2 \leq 1-\eta_\nu^2$ therefore using lemma 4.4.1, we have

$$(4.6.11) \qquad u_1 \leq M_0 \sum_{\nu=1}^{n} \frac{|x_\nu|}{(1-x_\nu^2)^{1/2}}\, \ell_\nu^2(0) + c_{23}\, n^{-1/2} .$$

Now consider

$$\sum_{\nu=1}^{n} \frac{|x_\nu|}{(1-x_\nu^2)^{1/2}}\, \ell_\nu^2(0) = |\omega_n(0)| \sum_{\nu=1}^{n} \frac{|\ell_\nu(0)|}{(1-x_\nu^2)^{1/2}|\omega'_n(x_\nu)|}$$

Applying Schwarz's inequality we have

$$\sum_{\nu=1}^{n} \frac{|x_\nu| \ell_\nu^2(0)}{(1-x_\nu^2)^{1/2}} \leq |\omega_n(0)| \left[\sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)[\omega_n'(x_\nu)]^2} \right]^{1/2} \left[\sum_{\nu=1}^{n} \ell_\nu^2(0) \right]^{1/2} .$$

With the help of (3.3.6), lemma 4.4.3 and (4.4.13) we have

$$\sum_{\nu=1}^{n} \frac{|x_\nu|}{(1-x_\nu^2)^{1/2}} \ell_\nu^2(0) \leq c_{24}\, n^{-1/2} . \tag{4.6.12}$$

Hence from (4.6.10), (4.6.11) and (4.6.12) we have

$$|d_n\, \omega_n(x)| \leq c_{25}\, n^{-1/2} . \tag{4.6.13}$$

so from (4.6.8) and (4.6.13) we have

$$|f(x) - R_n(x;f)| \leq c_{21} \log n\, [n^{-\mu} + n^{-1/4}] + c_{25}\, n^{-1/2}$$

$$= o(1) .$$

This completes the proof of Theorem 4.2.2.

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